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Admissible disturbances for perturbed nonlinear discrete systems

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Consider the discrete perturbed controlled nonlinear system given by

$$\begin{cases} x^e(i+1) = Ax^e(i) + f(\zeta_i u_i + \omega_i), \quad i \geqslant 0 \\ x^e(0) = \gamma x_0 + \psi \end{cases}$$

and the output function $y^e(i) = Cx^e(i),\ i \geqslant 0$, where $e = (\gamma, \psi, (\zeta_i)_{i\geqslant 0}, (\omega_i)_{i\geqslant 0})$ is a disturbance which perturb the system. The disturbance e is said to be ε-admissible if $\|y^e(i) - y(i)\| \leqslant \varepsilon$, $\forall i \geqslant 0$, where $(y(i))_{i\geqslant 0}$ is the output signal corresponding to the uninfected system. The set of all ε- admissible disturbances is the admissible set $\mathcal{S}(\varepsilon)$. The characterization of $\mathcal{S}(\varepsilon)$ is investigated and practical algorithms with numerical simulation are given. The admissible set $\mathcal{S}_d(\varepsilon)$ for discrete delayed systems is also considered.

Key words: discrete nonlinear systems, disturbances, asymptotic stability, admissibility, observability, discrete delayed systems

1. Introduction

During the control of a system, we are always confronted to the presence of some undesirable parameters. To better avoid damages being able to be caused by such perturbation on the evolution of a system, many approach have been developed (see [1], [2], [4], [5], [6], [7], [8] and [10]). We contribute in this direction to fix a threshold of tolerance ε (ε is chosen in function of the considered system) and to characterize, on the theoretic and algorithmic plan, perturbations of which the effect is under this threshold. The linear case has been dealt by Rachik and al in [9], we have also dealt the bilinear case (see [3]). So, as a natural continuation of what has been done [9] and [3], we devote this paper to the study of nonlinear case. Moreover to the difference of what has been done in [9] and [3], we suppose in this work that the initial state of the system is also affected by a perturbation.

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In a precise manner, we consider here the controlled perturbed nonlinear system defined by

$$\begin{cases} x^e(i+1) = Ax^e(i) + f(\zeta_i u_i + \omega_i) \ , \ i \geqslant 0 \\ x^e(0) = \gamma x_0 + \psi \end{cases} \tag{1}$$

the corresponding output signal is

$$y^e(i) = Cx^e(i) \ , \ i \geqslant 0 \tag{2}$$

where A, C are respectively $n \times n, p \times n$ matrices; $x^e(i) \in \mathbb{R}^n$ is the state variable, $u_i \in \mathbb{R}^m$ is the control variable, $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a continuous function, $e = (\gamma, \psi, (\zeta_i)_{i\geqslant 0}, (\omega_i)_{i\geqslant 0})$ is an undesirable disturbance which affects the system because of it's connections with the environment. The output signal corresponding to $\gamma = \zeta_i = 1$, $\psi = \omega_i = 0$ for all $i \in \mathbb{N}$ is simply denoted by $(y(i))_{i\geqslant 0}$, i.e.

$$y(i) = Cx(i), \quad i \geqslant 0 \tag{3}$$

where $(x(i))_{i\geqslant 0}$ is the uninfected state given by

$$\begin{cases} x(i+1) = Ax(i) + f(u_i) \ , \ i \geqslant 0 \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases} \tag{4}$$

In all of this paper, we suppose that the disturbances $(\omega_i)_{i\geqslant 0}$ and $(\zeta_i)_{i\geqslant 0}$ susceptible of infecting our system have a limited age, consequently in all this work we suppose that

$$\omega = (\omega_i)_{i\geqslant 0} \in \mathcal{U}_m^I = \{(\gamma_i)_{i\geqslant 0} / \gamma_i \in \mathbb{R}^m, \text{ and } \gamma_i = 0, \quad \forall i \geqslant I\}$$

and

$$\zeta = (\zeta_i)_{i\geqslant 0} \in \mathcal{R}_1^J = \{(\lambda_i)_{i\geqslant 0} / \lambda_i \in \mathbb{R}, \text{ and } \lambda_i = 1, \quad \forall i \geqslant J\}$$

where I and J are respectively the ages of disturbances $(\omega_i)_i$ and $(\zeta_i)_i$.

The ε-admissible set $S(\varepsilon)$ defined by

$$S(\varepsilon) = \{e = (\gamma, \psi, \omega, \zeta) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{U}_m^I \times \mathcal{R}_1^J / \ \|y^e(i) - y(i)\| \leqslant \varepsilon, \quad \forall i \geqslant 0\} \tag{5}$$

A summary of the contents of the paper reads as follows: In section 2, after introducing the notations we will use in the continuation of the paper, we give the propreties to characterize the set by functional inequalities and the condition under which $S(\varepsilon)$ contains a neighborhood of zero. A condition for finite determinability and an algorithmic procedure for the computation of $S(\varepsilon)$ are given in section 3. To illustrate this, we give some examples in section 4. Section 5 is devoted to the study of the characterization of admissible disturbances for discrete delayed systems.

Example 1 As an example of the characterization of ε-disturbances we give the following motivation. Let consider the discrete system described by

$$\begin{aligned} x^e_{i+1} &= 0.2x_i + (u_i + \omega_i)^2, \quad \forall i \geqslant 0, \\ x^e_0 &= x_0 + \psi, \end{aligned}$$

where x_i, u_i in $\mathbb{R}$, $I = 0$ the age of perturbation and the out put state is given by

$$y^e_i = x^e_i.$$

The ε-admissible set $S(\varepsilon)$ is defined by

$$S(\varepsilon) = \{e = (\psi, \omega) \in \mathbb{R}^2 / \ |y^e(i) - y(i)| \leqslant \varepsilon, \quad \forall i \geqslant 0\}$$

where the output signal corresponding to $e = (\psi, \omega) = 0$ is simply denoted by $(y(i))_{i\geqslant 0}$

$$y_i = x_i \quad , \ i \geqslant 0 \tag{6}$$

and $(x_i)_{i\geqslant 0}$ is the uninfected state given by

$$\begin{cases} x_{i+1} = 0.2x_i + u_i^2 \ , \ i \geqslant 0 \\ x(0) = x_0 \in \mathbb{R}. \end{cases} \tag{7}$$

We deduce by simple calculous that

$$S(\varepsilon) = \{e = (\psi, \omega) \in \mathbb{R}^2 / \ |\psi| \leqslant \varepsilon, |0.2\psi + (\omega_0 + u_0)^2 - u_0^2| \leqslant \varepsilon 2^{i-1}, \quad \forall i \geqslant 1\}.$$

If we take $u_0 = 0.2$ and $\varepsilon = 10^{-2}$ then, we deduce by

$$\begin{aligned} &\{(\psi, \omega) \in \mathbb{R}^2 / \ |0.2\psi + (\omega_0 + 0.2)^2 - 0.04| \leqslant 10^{-2}\} \subset \\ &\{(\psi, \omega) \in \mathbb{R}^2 / \ |0.2\psi + (\omega_0 + 0.2)^2 - 0.04| \leqslant 10^{-2} 2^{i-1}, \ \forall i \geqslant 1\} \end{aligned}$$

that

$$S(\varepsilon) = \{e = (\psi, \omega) \in \mathbb{R}^2 / \ |\psi| \leqslant 10^{-2}, |0.2\psi + (\omega_0 + 0.2)^2 - 0.04| \leqslant 10^{-2}\}.$$

The admissible set corresponding to previous system is given in Fig. 1.

2. Preliminary results

It is easy to deduce from equation (1) that

$$x^e(i) = A^i x^e(0) + \sum_{j=0}^{i-1} A^{i-1-j} f(\zeta_j u_j + \omega_j) \ , \forall i \geqslant 0$$

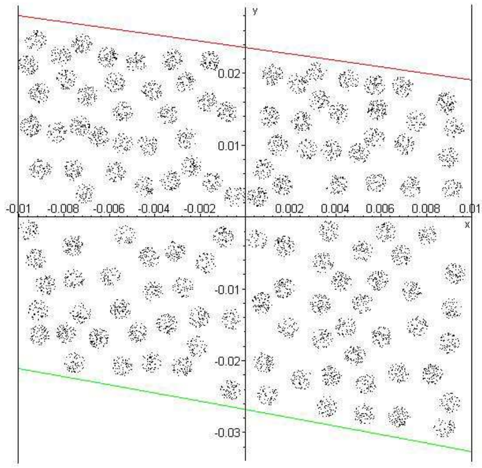

Figure 1. The set $\mathcal{S}(\varepsilon)$ corresponding to example 1

then

$$y^e(0) - y(0) = (\gamma - 1)Cx(0) + C\psi$$

and for every $i \geqslant 1$, we have

$$\begin{aligned} y^e(i) - y(i) &= Cx^e(i) - Cx(i) \\ &= CA^i((\gamma-1)x(0)+\psi) + \sum_{j=0}^{i-1} CA^{i-j-1}(f(\zeta_j u_j + \omega_j) - f(u_j)). \end{aligned}$$

If we introduce the signal $(\xi_j^e)_{j\geqslant 0} \in \mathcal{U}_n^{\max(I,J)+1}$ defined by

$$\begin{cases} \xi_{j+1}^e = f(\zeta_j u_j + \omega_j) - f(u_i), \quad \forall j \geqslant 0 \\ \xi_0^e = (\gamma - 1)x(0) + \psi \end{cases} \tag{8}$$

we easily establish that

$$y^e(i) - y(i) = \sum_{j=0}^{i} CA^{i-j}\xi_j^e \ .$$

Consequently, the set $S(\varepsilon)$ of all disturbances $e = (\gamma, \psi, \zeta, \omega) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{R}_1^J \times \mathcal{U}_m^I$ can be written as follows

$$S(\varepsilon) = \{e = (\gamma, \psi, \zeta, \omega) / \| \sum_{j=o}^{i} CA^{i-j}\xi_j^e \| \leqslant \varepsilon \ , \ \forall i \geqslant 0\}.$$

Since $\mathcal{U}_r^s$ (Resp $\mathcal{R}_r^s$) can be identified to $\mathbb{R}^{r(s+1)}$ by the canonical isomorphism

$$\begin{array}{rccc} \varphi: & \mathcal{U}_r^s & \longrightarrow & \mathbb{R}^{r(s+1)} \\ & (z_i)_{i\geqslant 0} & \longrightarrow & (z_i)_{i\leqslant s}^{\top} \end{array}$$

$$\left(\begin{array}{llll} \text{Resp} & \varphi : & \mathcal{R}_r^s & \longrightarrow \mathbb{R}^{r(s+1)} \\ & & (z_i)_{i\geqslant 0} & \longrightarrow (z_i)_{i\leqslant s}^{\top} \end{array} \right)$$

where $(z_i)_{i\leqslant s}^{\top}$ is the vector of $\mathbb{R}^{r(s+1)}$ given by

$$(z_i)_{i\leqslant s} = \begin{bmatrix} z_0 \\ \vdots \\ z_s \end{bmatrix}^{\top} \in \underbrace{\mathbb{R}^r \times \mathbb{R}^r \ldots \mathbb{R}^r}_{s+1-times}$$

then

$$S(\varepsilon) = \{e = (\gamma,\psi,\zeta,\omega) \in \mathcal{M} \ / \ \|\sum_{j=o}^{i} CA^{i-j}\xi_j^e\| \leqslant \varepsilon, \quad \forall i \geqslant 0\}$$

with

$$\mathcal{M} = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{J+1} \times \mathbb{R}^{m(I+1)}. \tag{9}$$

In order to characterize the set $S(\varepsilon)$ by a finite number of functional inequalities, we rewrite $S(\varepsilon)$ as follows

$$S(\varepsilon) = \mathcal{V}(\varepsilon) \cap \mathcal{W}(\varepsilon) \tag{10}$$

where

$$\mathcal{V}(\varepsilon) = \{e \in \mathcal{M} \ /\|\sum_{j=o}^{i} CA^{i-j}\xi_j^e\| \leqslant \varepsilon \ , \ \forall i \in \{0,\ldots,\max(I,J)\}\} \tag{11}$$

and

$$\mathcal{W}(\varepsilon) = \{e \in \mathcal{M} \ /\|\sum_{j=o}^{i} CA^{i-j}\xi_j^e\| \leqslant \varepsilon \ , \ \forall i \geqslant \max(I,J)+1\}. \tag{12}$$

Since the set $\mathcal{V}(\varepsilon)$ is characterized by a finite number of inequalities, our objective will be the characterization of the set $\mathcal{W}(\varepsilon)$. We have

$$\begin{aligned} \mathcal{W}(\varepsilon) &= \{e \in \mathcal{M} \ / \ \|C \sum_{j=0}^{\max(I,J)} A^{i-j}\xi_j^e\| \leqslant \varepsilon, \quad \forall i \geqslant \max(I,J)+1\} \\ &= \{e \in \mathcal{M} \ / \ \|CA^{k+1} \sum_{j=0}^{\max(I,J)} A^{\max(I,J)-j}\xi_j^e\| \leqslant \varepsilon \ , \quad \forall k \geqslant 0\} \\ &= \{e \in \mathcal{M} \ / \ \|CA^{k+1}\mathcal{G}(e)\| \leqslant \varepsilon \ , \quad \forall k \geqslant 0\} \end{aligned}$$

where $\mathcal{G}$ is the map defined by

$$\begin{array}{llll} \mathcal{G} : & \mathbb{R} \times \mathbb{R}^n \times \overbrace{\mathbb{R} \times \ldots \times \mathbb{R}}^{(J+1)-times} \times \overbrace{\mathbb{R}^m \times \ldots \times \mathbb{R}^m}^{(I+1)-times} & \longrightarrow & \mathbb{R}^n \\ & e = (y,z,y_0,\ldots,y_J,z_0,\ldots,z_I) & \longmapsto & \sum_{j=0}^{\max(I,J)} A^{\max(I,J)-j}\xi_j^e \end{array} \tag{13}$$

with $(\xi_j^e)_{j\geqslant 0}$ given by

$$\begin{cases} \xi_{j+1}^e &= f(y_j u_j + z_j) - f(u_j) \\ \xi_0^e &= (y-1)x(0) + z. \end{cases}$$

Proposition 1 *If A is Lyapunov stable(the characteristic roots of A satisfy the following conditions :* $|\lambda| \leqslant 1$ *for every* λ *in the spectrum of A, and* $|\lambda| = 1$ *implies* λ *is simple) then* $0_1 \in int\,\mathcal{S}(\varepsilon)$*, where* $0_1 = ((1, 0_n, 1, \dots, 1, 0_m, \dots, 0_m)) \in \mathcal{M}$ *,* 0_n *and* 0_m *are the* $n\times n$*-zero and* $m\times m$*-zero matrix respectively.*

PROOF. We have

$$\mathcal{V}(\varepsilon) = \bigcap_{i=0}^{\max(I,J)} \mathcal{V}_i(\varepsilon) \supset \bigcap_{i=0}^{\max(I,J)} \widehat{\mathcal{V}_i}(\varepsilon)$$

where

$$\mathcal{V}_i(\varepsilon) = \{e \in \mathcal{M} \ \ / \ \| \sum_{j=o}^{\max(I,J)} CA^{i-j}\xi_j^e \| \leqslant \varepsilon\}$$

and

$$\widehat{\mathcal{V}_i}(\varepsilon) = \{e \in \mathcal{M} \ \ / \ \| \sum_{j=o}^{\max(I,J)} CA^{i-j}\xi_j^e \| < \varepsilon\}.$$

Moreover, we use the continuity of f to deduce that the map

$$e \in \mathcal{M} \hookrightarrow \| \sum_{j=o}^{\max(I,J)} CA^{i-j}\xi_j^e \|$$

is continuous too.

Consequently $\widehat{\mathcal{V}_i}(\varepsilon)$ is an open subset of $\mathcal{M} = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{I+1} \times \mathbb{R}^{m(J+1)}$ which contains the value 0_1, thus $0_1 \in \mathrm{int}\mathcal{V}(\varepsilon)$. On the other hand the Lyapunov stability of A implies the existence of a constant $\gamma > 0$ such that

$$\|CA^{k+1}x\| \leqslant \gamma\|x\| \ \text{ for every } \ x \in \mathbb{R}^n \ \text{ and } \ k \in \mathbb{N}.$$

For every $(t,x,y,z) \in \mathcal{M}$ and every $k \in \mathbb{N}$ we have

$$\|CA^{k+1}\mathcal{G}(t,x,y,z)\| \leqslant \gamma\|\mathcal{G}(t,x,y,z)\|.$$

Moreover the continuity of $\mathcal{G}$ implies that

$$\forall\varepsilon > 0 \ \ \exists\eta > 0, \ \ \|(t,x,y,z) - 0_1\| \leqslant \eta \Rightarrow \|\mathcal{G}(t,x,y,z)\| \leqslant \varepsilon/\gamma$$

so for every $(t,x,y,z) \in \mathcal{B}_{\mathcal{M}}(0_1,\eta)$ (where $\mathcal{B}_{\mathcal{M}}(0_1,\eta)$ is ball of 0_1 in centre and η radius) and every $k \in \mathbb{N}$ we have

$$\|CA^{k+1}\mathcal{G}(t,x,y,z)\| \leqslant \gamma\|\mathcal{G}(t,x,y,z)\| \leqslant \varepsilon,$$

hence $\mathcal{B}_{\mathcal{M}}(0_1,\eta) \subset \mathcal{W}(\varepsilon)$ thus $0_1 \in \mathrm{int}\mathcal{W}(\varepsilon)$. □

3. The characterization of $\mathcal{W}(\varepsilon)$

In order to characterize the set $\mathcal{W}(\varepsilon)$ by a finite number of inequalities, we rewrite it as follows

$$\mathcal{W}(\varepsilon) = \{e = (\gamma, \psi, (\zeta_i)_{i\leqslant J}, (\omega_i)_{i\leqslant I}) \in \mathcal{M} \quad / \quad \mathcal{G}(e) \in \mathcal{T}(\varepsilon)\} \tag{14}$$

where

$$\mathcal{T}(\varepsilon) = \{x \in \mathbb{R}^n / \|CA^{i+1}x\| \leqslant \varepsilon \quad , \quad \forall i \geqslant 0\}. \tag{15}$$

For every $k \in \mathbb{N}$, we define the set $\mathcal{T}_k(\varepsilon)$ by

$$\mathcal{T}_k(\varepsilon) = \{x \in \mathbb{R}^n / \|CA^{i+1}x\| \leqslant \varepsilon \quad \forall i \in \{0, 1, \ldots, k\}\}, \tag{16}$$

$\mathcal{T}(\varepsilon)$ is said to be finitely accessible if there exists $k \in \mathbb{N}$ such that $\mathcal{T}(\varepsilon) = \mathcal{T}_k(\varepsilon)$, we note k^* the smallest integer such that $\mathcal{T}(\varepsilon) = \mathcal{T}_{k^*}(\varepsilon)$.

Remark 1 *We have*

$$\mathcal{T}(\varepsilon) \subset \mathcal{T}_{k_2}(\varepsilon) \subset \mathcal{T}_{k_1}(\varepsilon) \quad , \quad \forall k_1, k_2 \in \mathbb{N} \textit{ such that } k_1 \leqslant k_2. \tag{17}$$

Proposition 2 $\mathcal{T}(\varepsilon)$ *is finitely accessible if and only if* $\mathcal{T}_{i+1}(\varepsilon) = \mathcal{T}_i(\varepsilon)$ *for some* $i \in \mathbb{N}$

PROOF. If $\mathcal{T}(\varepsilon)$ is finitely accessible, then the equality holds for all $i \geqslant k^*$. Conversely, if $\mathcal{T}_{i+1}(\varepsilon) = \mathcal{T}_i(\varepsilon)$ for some $i \in \mathbb{N}$, we deduce that $\mathcal{T}_i(\varepsilon)$ is A-invariant (i.e. $A(\mathcal{T}_i(\varepsilon)) \subset \mathcal{T}_i(\varepsilon)$) which implies that $\mathcal{T}_i(\varepsilon)$ is A^k- invariant for every $k \in \mathbb{N}$, and so $\mathcal{T}_i(\varepsilon) \subset \mathcal{T}(\varepsilon)$, finally we apply Remark 1 to end the proof. □

Using proposition 1 we can establish a first formal algorithm to determine the smallest integer $k^\star$ such that $\mathcal{T}_{k^\star}(\varepsilon) = \mathcal{T}(\varepsilon)$ and consequently to characterize the set $\mathcal{W}(\varepsilon)$ by

$$\mathcal{W}(\varepsilon) = \mathcal{W}_{k^\star}(\varepsilon) = \mathcal{G}^{-1}(\mathcal{T}_{k^\star}(\varepsilon)).$$

Algorithm 1

step 1 : Set $k = 0$

step 2 : If $\mathcal{T}_{k+1}(\varepsilon) = \mathcal{T}_k(\varepsilon)$ then set $k^* = k$ and stop,
else continue.

step 3 : Replace k by $k+1$ and return to step 2.

It is obvious that algorithm I is not practical because it does not describe how the test $\mathcal{T}_{k+1}(\varepsilon) = \mathcal{T}_k(\varepsilon)$ is implemented, moreover it produces k^* if and only if $\mathcal{T}(\varepsilon)$ is finitely accessible. In order to overcome this difficulty, let $\mathbb{R}^n$ be endowed with the following norm

$$\|x\| = \max_{1\leqslant i\leqslant n} |x_i| \quad , \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

The set $\mathcal{T}_k(\varepsilon)$ is then described as follows

$$\mathcal{T}_k(\varepsilon) = \{x \in \mathbb{R}^n / h_j(CA^{i+1}x) \leqslant 0 \text{ for } j = 1,2,\ldots,2p \text{ and } i = 0,1,\ldots,k\} \tag{18}$$

where $h_j : \mathbb{R}^p \longrightarrow \mathbb{R}$, are defined for every $x = (x_1,\ldots,x_p) \in \mathbb{R}^p$ by

$$h_{2m-1}(x) = x_m - \varepsilon, \text{ for } m \in \{1,2,\ldots p\},$$

$$h_{2m}(x) = -x_m - \varepsilon, \text{ for } m \in \{1,2,\ldots p\}.$$

It follows from remark 1 that

$$\mathcal{T}_{k+1}(\varepsilon) = \mathcal{T}_k(\varepsilon) \Longleftrightarrow \mathcal{T}_k(\varepsilon) \subset \mathcal{T}_{k+1}(\varepsilon)$$

so

$$\mathcal{T}_{k+1}(\varepsilon) = \mathcal{T}_k(\varepsilon) \Longleftrightarrow [\forall x \in \mathcal{T}_k(\varepsilon),\ \forall j \in \{1,2,\ldots,2p\} \quad h_j(CA^{k+2}x) \leqslant 0] \tag{19}$$

or equivalently

$$\sup_{x \in \mathcal{T}_k(\varepsilon)} h_j(CA^{k+2}x) \leqslant 0 \quad \forall j \in \{1,2,\ldots,2p\}, \tag{20}$$

hence algorithm 1 can be rewritten as follows.

Algorithm 2

step 1 : Let $k = 0$;

step 2 : For $i = 1,\ldots,2p$, do :

  Maximize $J_i(x) = h_i(CA^{k+2}x)$
  $$\begin{cases} h_i(CA^l x) \leqslant 0, \\ i = 1,\ldots,2p,\ l = 1,\ldots,k+1. \end{cases}$$

  Let J_i^* be the maximum value of $J_i(x)$.

  If $J_i^* \leqslant 0$, for $i = 1,\ldots,2p$ then

    set $k^* := k$ and stop.

  Else continue.

step 3 : Replace k by $k+1$ and return to step 2.

Remark 2 *The optimization problem cited in step* 2 *is a mathematical programming problem and can be solved by standard methods.*

It is clear that algorithm II converges if and only if there exists an integer k such that $\mathcal{T}_{k+1}(\varepsilon) = \mathcal{T}_k(\varepsilon)$, so it is desirable to establish simple conditions which make the set $\mathcal{W}(\varepsilon)$ (or $\mathcal{T}(\varepsilon)$) finitely accessible. Our main result in this direction is the following theorem.

Theorem 1 *Suppose the following assumptions to hold*
i) A is asymptotically stable($|\lambda| < 1$ for every λ in spectrum of A),
ii) the pair (C,A) is observable $([C^\top|A^\top C^\top|\dots|(A^\top)^{n-1}C^\top]$ has rank n).
Then $\mathcal{W}(\varepsilon)$ is finitely accessible.

PROOF. Let $x \in \mathcal{T}_{n-1}(\varepsilon)$ then $\|CA^{i+1}x\| \leqslant \varepsilon \quad \forall i \in \{0,1,\dots,n-1\}$ which implies that

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} Ax \in \overbrace{\mathcal{B}_p(0,\varepsilon) \times \dots \times \mathcal{B}_p(0,\varepsilon)}^{n-times}$$

where

$$\mathcal{B}_p(0,\varepsilon) = \{x \in \mathbb{R}^p / \|x\| \leqslant \varepsilon\}.$$

Hence $\Lambda^\top \Lambda A x \in \Lambda^\top(\overbrace{\mathcal{B}_p(0,\varepsilon) \times \dots \times \mathcal{B}_p(0,\varepsilon)}^{n-times})$ where Λ is the matrix given by

$$\Lambda = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{np}).$$

Consequently

$$(\Lambda^\top \Lambda A)(T_{n-1}(\varepsilon)) \subset \Lambda^\top(\overbrace{\mathcal{B}_p(0,\varepsilon) \times \dots \times \mathcal{B}_p(0,\varepsilon)}^{n-times}). \tag{21}$$

So, for every $x \in \mathcal{T}_{n-1}(\varepsilon)$ there exists $z \in (\overbrace{\mathcal{B}_p(0,\varepsilon) \times \dots \times \mathcal{B}_p(0,\varepsilon)}^{n-times})$ such that $\Lambda^\top \Lambda A x = \Lambda^\top z$, which implies that

$$< \Lambda^\top \Lambda A x, Ax > \; = \; < \Lambda^\top z, Ax > \; , \quad \forall x \in \mathcal{T}_{n-1}(\varepsilon). \tag{22}$$

On the other hand the obsevability of (A,C) implies that $\Lambda^\top \Lambda$ is coercive i.e.

$$\exists \alpha > 0 \;\; / \;\; < \Lambda^\top \Lambda x, x > \; \geqslant \; \alpha \|x\|^2 \;\; , \;\; \forall x \in \mathbb{R}^n,$$

then it follows from (22) that

$$\alpha \|Ax\|^2 \leqslant (cste)\|Ax\| \times \|z\|, \quad \forall x \in \mathcal{T}_{n-1}(\varepsilon)$$

and consequently

$$\|Ax\| \leqslant (cste)\|z\| \quad , \quad \forall x \in \mathcal{T}_{n-1}(\varepsilon).$$

Then, since $(\overbrace{\mathcal{B}_p(0,\varepsilon)\times\ldots\times\mathcal{B}_p(0,\varepsilon)}^{n-times})$ is a bounded set, we deduce the existence of a constant $r>0$ such that

$$A\mathcal{T}_k(\varepsilon)\subset\mathcal{B}_n(0,r)=\{x\in\mathbb{R}^n/\|x\|\leqslant r\},\quad\forall k\geqslant n-1.$$

Using the asymptotic stability of A, it follows that there exists $k_0\geqslant n-1$ such that $\|CA^{k_0+1}\|\leqslant\dfrac{\varepsilon}{r}$, hence

$$CA^{k_0+1}(\mathcal{B}_n(0,r))\subset\mathcal{B}_p(0,\varepsilon)$$

then

$$\|CA^{k_o+2}x\|\leqslant\varepsilon\quad\forall x\in T_{k_0}(x)$$

which implies that

$$x\in\mathcal{T}_{k_0+1}(\varepsilon).$$

Consequently

$$\mathcal{T}_{k_0}(\varepsilon)\subset\mathcal{T}_{k_0+1}(\varepsilon)$$

Finally, we use proposition 2 to end the proof. □

4. Examples

In this section, we give two simple examples where we present the set $\mathcal{S}(\varepsilon)$.

Example 2 Let A, C and ε given by

$$A=\begin{pmatrix}0.6&0\\1&0.7\end{pmatrix},\quad C=\begin{pmatrix}2&0\\1&1\end{pmatrix}\quad\varepsilon=1.$$

Then, we use algorithm 2 to establish that $k^*=3$.
We suppose that $f:\ \mathbb{R}\longrightarrow\mathbb{R}^2:x\hookrightarrow(0,x+1)$, $\omega_i=0,\ \forall i\geqslant1,\ \zeta_i=1,\ \forall i\geqslant0$ and $\gamma=1$. Then for all $u_0\in\mathbb{R}$, we have

$$\mathcal{S}(1)=\{(\psi_1,\psi_2,\omega_0)\in\mathbb{R}^3/\ |\psi_1|\leqslant1\ \ |\psi_1+\psi_2|\leqslant1,\ \ |-0.8\psi_1+0.5\psi_2+\omega_0+1|\leqslant1\}.$$

Example 3 For $A=1$, $C=1$, and $\varepsilon=0.01$, we obtain $k^*=1$. If we take $f:\mathbb{R}\longrightarrow\mathbb{R}:x\hookrightarrow x^2$, $\omega_i=0,\ \forall i\geqslant2,\ \zeta_i=1,\ \forall i\geqslant0$ and $\gamma=1$, then for $u_0=0$ and $u_1=1$, we have

$$\mathcal{S}(\varepsilon)=\{(\psi,\omega_0,\omega_1)\in\mathbb{R}^3/\ |\psi|\leqslant\varepsilon,\ |\psi+\omega_0^2|\leqslant\varepsilon,\ |\psi+\omega_0^2+(\omega_1+1)^2-1|\leqslant\varepsilon\}.$$

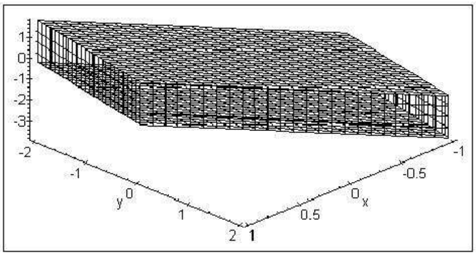

Figure 2. The set $\mathcal{S}(\varepsilon)$ corresponding to example 2

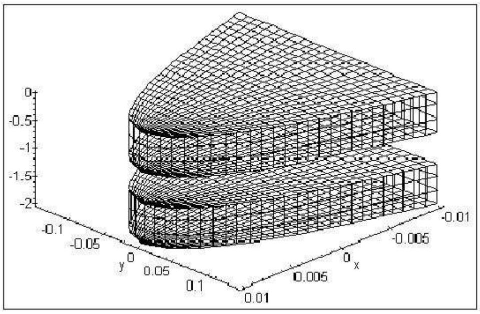

Figure 3. The set $\mathcal{S}(\varepsilon)$ corresponding to example 3

5. Admissible disturbances for discrete delayed nonlinear systems

This section is devoted to the characterization of admissible disturbances for the discrete delayed system given by

$$\begin{cases} x^e(i+1) &= \sum_{j=0}^{r} A_j x^e(i-j) + f(\zeta_i u_i + \omega_i)\,, \quad i \geqslant 0 \\ x^e(k) &= \gamma_k \theta_k + \psi_k \quad \text{for} \quad k \in \{-r, -r+1, \ldots, -1, 0\}. \end{cases} \tag{23}$$

The corresponding delayed output function is

$$y^e(i) = \sum_{j=0}^{s} C_j x^e(i-j)\,, \quad i \geqslant 0 \tag{24}$$

where $A_j \in \mathcal{L}(\mathbb{R}^n), C_j \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, r and s are integers such that $s \leqslant r$, $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a continuous function, $\Psi = (\psi_{-r}, \psi_{-r+1}, \ldots, \psi_{-1}, \psi_0)^\top \in (\mathbb{R}^n)^{r+1}$ and $\Gamma = (\gamma_{-r}, \gamma_{-r+1}, \ldots, \gamma_{-1}, \gamma_0)^\top \in \mathbb{R}^{r+1}$ are a perturbations which infect the initial state $\theta = (\theta_{-r}, \theta_{-r+1}, \ldots, \theta_{-1}, \theta_0)^\top$.

As before we suppose that I and J are respectively the ages of $\zeta = (\zeta_i)_{i \geqslant 0}$ and $\omega = (\omega_i)_{i \geqslant 0}$ and we investigate the set $S_d(\varepsilon)$ of all ε-admissible disturbances

$$e = (\Gamma, \Psi, \zeta, \omega) \in \mathbb{R}^{r+1} \times (\mathbb{R}^n)^{r+1} \times \mathcal{R}_1^J \times \mathcal{U}_m^I$$

i.e.

$$S_d(\varepsilon) = \{e \in \mathbb{R}^{r+1} \times (\mathbb{R}^n)^{r+1} \times \mathcal{R}_1^J \times \mathcal{U}_m^I / \|y^e(i) - y(i)\| \leqslant \varepsilon, \ \forall i \geqslant 0\}$$

where $(y(i))_{i \geqslant 0}$ is the output function corresponding to the uninfected controlled system, that is

$$y(i) = \sum_{j=0}^{s} C_j x(i-j) \,, \quad i \geqslant 0 \tag{25}$$

with

$$\begin{cases} x(i+1) &= \sum_{j=0}^{r} A_j x(i-j) + f(u_i) \,, \quad i \geqslant 0 \\ x(k) &= \theta_k \ \text{ for } \ k \in \{-r, -r+1, \ldots, -1, 0\}. \end{cases} \tag{26}$$

Consider the new state variables $X^e(i)$ and $X(i)$ defined in $\mathbb{R}^{n(r+1)}$ by

$$X^e(i) = (x^e(i), x^e(i-1), \ldots, x^e(i-r))^\top, \qquad i \geqslant 0$$

$$X(i) = (x(i), x(i-1), \ldots, x(i-r))^\top, \qquad i \geqslant 0$$

and define the matrix $\widetilde{A}$ by

$$\widetilde{A} = \begin{pmatrix} A_0 & A_1 & \ldots & \ldots & A_r \\ I_n & 0_n & \ldots & \ldots & 0_n \\ 0_n & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_n & \ldots & 0_n & I_n & 0_n \end{pmatrix} \in \mathcal{L}(\mathbb{R}^{n(r+1)})$$

where I_n is the $n \times n$-unit matrix, O_n is the $n \times n$-zero matrix. Then it is easy to deduce from (23) and (26) that

$$\begin{cases} X^e(i+1) &= \widetilde{A} X^e(i) + F(\zeta u_i + \omega_i) \\ X^e(0) &= \Gamma \theta_0 + \Psi \end{cases} \tag{27}$$

and

$$\begin{cases} X(i+1) &= \widetilde{A} X(i) + F(u_i) \\ X(0) &= \theta_0 \end{cases} \tag{28}$$

where

$$\begin{aligned} F \quad : \quad & \mathbb{R}^m \longrightarrow \mathbb{R}^{n(r+1)} \\ & x \longrightarrow F(x) = (f(x), 0, \ldots, 0)^\top \end{aligned}$$

and

$$\Gamma \theta_0 = (\gamma_0 \theta_0, \gamma_{-1} \theta_{-1}, \ldots, \gamma_{-r} \theta_{-r})^\top.$$

Moreover, if we consider the matrix

$$\widetilde{C} = (C_0|C_1|\ldots|C_s|O_{p\times n}|\ldots|O_{p\times n}) \in \mathcal{L}(\mathbb{R}^{n(r+1)}, \mathbb{R}^p)$$

then, (24) and (25) are given in terms of the new state variables $X^e(i)$ and $X(i)$ by

$$y^e(i) = \widetilde{C}X^e(i), \qquad \forall i \geqslant 0, \tag{29}$$

$$y(i) = \widetilde{C}X(i), \qquad \forall i \geqslant 0. \tag{30}$$

Consequently, the set of the ε-admissible disturbances $e = (\Gamma, \psi, \zeta, \omega)$ is

$$S_d(\varepsilon) = \{e \in \mathcal{N} \ / \ \|\widetilde{C}\sum_{j=0}^{i-1}\widetilde{A}^{i-j}\widetilde{\xi^e}_j\| \leqslant \varepsilon, \ \ \forall i \geqslant 0\}$$

where

$$\mathcal{N} = \mathbb{R}^{r+1} \times \mathbb{R}^{n(r+1)} \times \mathbb{R}^{m(I+1)} \times \mathbb{R}^{q(J+1)}$$

and

$$\begin{cases} \widetilde{\xi}^e_{i+1} &= F(\zeta_i u_i + \omega_i) - F(u_i) \qquad \forall i \geqslant 0 \\ \widetilde{\xi}^e_0 &= (\Gamma\theta - \theta) + \Psi. \end{cases} \tag{31}$$

As previously, we have

$$S_d(\varepsilon) = V_d(\varepsilon)\bigcap W_d(\varepsilon)$$

where

$$V_d(\varepsilon) = \{e \in \mathcal{N} \ / \ \|\widetilde{C}\sum_{j=0}^{i}\widetilde{A}^{i-j}\widetilde{\xi^e}_j\| \leqslant \varepsilon, \ \ \forall i \in \{0,1,\ldots,\max(I,J)\}\}$$

and

$$W_d(\varepsilon) = \{e \in \mathcal{N} \ / \ \|\widetilde{C}\sum_{j=0}^{i}\widetilde{A}^{i-j}\widetilde{\xi^e}_j\| \leqslant \varepsilon, \ \ \forall i \geqslant \max(I,J)+1\}.$$

As in section 2 we have

$$W_d(\varepsilon) = \{e = ((\omega_k)_{-r\leqslant k\leqslant 0}, (\alpha_i)_{i\leqslant I}, (\zeta_i)_{i\leqslant J}) \in \mathcal{N} \ / \ \|\widetilde{C}\widetilde{A}^{k+1}\widetilde{\mathcal{G}}(e)\| \leqslant \varepsilon \ \ , \forall k \geqslant 0\}$$

where $\widetilde{\mathcal{G}}$ is the map defined by

$$\begin{aligned} \widetilde{\mathcal{G}} \ : \ \mathcal{N} = \mathbb{R}^{n(r+1)} \times \mathbb{R}^{m(I+1)} \times \mathbb{R}^{q(J+1)} &\longrightarrow \mathbb{R}^{n(r+1)} \\ e = ((x_k)_{-r\leqslant k\leqslant 0}, (y_k)_{-r\leqslant k\leqslant 0}, (z_i)_{i\leqslant I}, (t_i)_{i\leqslant J}) &\longrightarrow \sum_{i=o}^{\max(I,J)} (\widetilde{A})^{\max(I,J)-i}\widetilde{\xi^e}_i \end{aligned}$$

with

$$\begin{cases} \widetilde{\xi^e}_{i+1} &= F(z_i u_i + t_i) - F(u_i) \quad \forall i \geqslant 0 \\ \widetilde{\xi^e}_0 &= ((x_0 - 1)\theta_0 + y_0, (x_{-1} - 1)\theta_{-1} + y_{-1}, \ldots, (x_{-r} - 1)\theta_{-r} + y_{-r})^\top. \end{cases}$$

In order to characterize $W_d(\varepsilon)$, we introduce the following sets

$$\mathcal{T}_d(\varepsilon) = \{x \in \mathbb{R}^{n(r+1)} / \|\widetilde{C}\widetilde{A}^{i+1}x\| \leqslant \varepsilon, \quad \forall i \geqslant 0\}, \tag{32}$$

$$\mathcal{T}_d^k(\varepsilon) = \{x \in \mathbb{R}^{n(r+1)} / \|\widetilde{C}\widetilde{A}^{i+1}x\| \leqslant \varepsilon \quad \forall i \in \{0, 1, \ldots, k\}\}, \quad k \geqslant 0, \tag{33}$$

and

$$W_d^k(\varepsilon) = \widetilde{\mathcal{G}}^{-1}(\mathcal{T}_d^k(\varepsilon)), \quad k \geqslant 0. \tag{34}$$

It is obvious that theorem 1 gives sufficient conditions to make $W_d(\varepsilon)$ finitely accessible. In the following we focus our interest in finding sufficient condition adapted to discrete delayed systems so that $W_d^k(\varepsilon)$ be finitely determined. In our study we consider two cases

a) First case, $p = n$ (i.e. the observation space and the state space have the same dimension).

b) Second case, $p < n$(which is the usual case.)

First case, $p = n$.

In this case every C_i is an $n \times n$ matrix.

Proposition 3 *Suppose the following assumptions to hold*

i) C_i commutes with A_j for all i and j such that $0 \leqslant i \leqslant s, \ 0 \leqslant j \leqslant r$

ii) $\|\sum_{i=0}^{r} A_i z_i\| \leqslant \varepsilon$ *for every* $(z_0, \ldots, z_r) \in \underbrace{\mathcal{B}_n(0,\varepsilon) \times \ldots \times \mathcal{B}_n(0,\varepsilon)}_{(r+1)-times}$

then $W_d(\varepsilon)$ is finitely determined, moreover $W_d(\varepsilon) = W_d^r(\varepsilon)$.

PROOF. Let $x = (x_0, x_1, \ldots, x_r) \in \mathbb{R}^{n(r+1)}$. If we set

$$h_k^x = \widetilde{C}\widetilde{A}^k x, \quad k \geqslant 0$$

then we have

$$h_k^x = \widetilde{C} Z_k^x, \quad k \geqslant 0 \tag{35}$$

where Z_k^x is such that

$$Z_k^x = (z_k, z_{k-1}, \ldots, z_{k-r}), \quad k \geqslant 0$$

and $(z_k)_{k\geqslant 0}$ the unique solution of the system

$$\begin{cases} z_{k+1} = \sum_{j=0}^{r} A_j z_{k-j} \\ z_{-i} = x_i \quad i \in \{0,\dots,r\}. \end{cases} \tag{36}$$

Indeed, from (36), we easily establish that

$$\begin{cases} Z^x_{k+1} = \widetilde{A} Z^x_k, \quad k \geqslant 0 \\ Z^x_0 = x \end{cases} \tag{37}$$

which implies that $Z^x_{k+1} = \widetilde{A}^{k+1}x$ and the equality (35). So h_k can be interpreted as the output function associated to system (37). Using (35) to deduce that

$$h^x_k = \sum_{j=0}^{s} C_j z_{k-j} \ , \quad k \geqslant 0$$

hence for $k \geqslant r+1$ we have

$$\begin{aligned} h^x_k &= \sum_{j=0}^{s} C_j z_{k-j} = \sum_{j=0}^{s} C_j \sum_{i=0}^{r} A_i z_{k-j-i-1} \\ &= \sum_{i=0}^{r} A_i \sum_{j=0}^{s} C_j z_{k-j-i-1} = \sum_{i=0}^{r} A_i h^x_{k-i-1}. \end{aligned} \tag{38}$$

Let $\overline{x} \in T^r_d(\varepsilon)$. We have

$$\|\widetilde{C}\widetilde{A}^{k+1}\overline{x}\| \leqslant \varepsilon, \quad \forall k \in \{0,1,\dots,r\}$$

or equivalently

$$\|h^{\overline{x}}_{k+1}\| \leqslant \varepsilon, \quad \forall k \in \{0,1,\dots,r\}. \tag{39}$$

Applying (38) for $k = r+2$, we have

$$h^{\overline{x}}_{r+2} = \sum_{i=0}^{r} A_i h^{\overline{x}}_{r+1-i}.$$

Using hypothesis ii) and (39), we deduce that

$$\|h^{\overline{x}}_{r+2}\| \leqslant \varepsilon$$

which implies that

$$\overline{x} \in T^{r+1}_d(\varepsilon)$$

and consequently

$$T^r_d(\varepsilon) \subset T^{r+1}_d(\varepsilon)$$

or

$$T_d(\varepsilon) = T_d^r(\varepsilon).$$

Hence

$$\widetilde{\mathcal{G}}^{-1}(T_d(\varepsilon)) = \widetilde{\mathcal{G}}^{-1}(T_d^r(\varepsilon)) \quad i.e. \quad W_d(\varepsilon) = W_d^r(\varepsilon).$$

□

Second case $p < n$.

Since every C_i is a $p \times n$ matrix, then $\hat{C}_i = \begin{pmatrix} C_i \\ 0 \end{pmatrix}$ is a $n \times n$-matrix. If we introduce the new observation variables $\hat{y}^e(i)$ and $\hat{y}(i)$ defined by

$$\hat{y}^e(i) = \begin{pmatrix} y^e(i) \\ 0_{\mathbb{R}^{n-p}} \end{pmatrix} \in \mathbb{R}^n, \qquad \hat{y}(i) = \begin{pmatrix} y(i) \\ 0_{\mathbb{R}^{n-p}} \end{pmatrix} \in \mathbb{R}^n$$

then clearly we have

$$\hat{y}^e(i) = \sum_{j=0}^{s} \hat{C}_j x^e(i-j), \quad \hat{y}(i) = \sum_{j=0}^{s} \hat{C}_j x(i-j). \tag{40}$$

Consequently the set $S_d(\varepsilon)$ is given by

$$\begin{aligned} S_d(\varepsilon) &= \{e \in \mathcal{N} / \|y^e(i) - y(i)\| \leqslant \varepsilon \;, \quad \forall i \geqslant 0\} \\ &= \{e \in \mathcal{N} / \|\hat{y}^e(i) - \hat{y}(i)\| \leqslant \varepsilon \;, \quad \forall i \geqslant 0\}. \end{aligned}$$

Finally, since $\hat{C}_i$ are $n \times n$-matrices, we apply the result established in the first case $(p = n)$, to the systems (23) and (26) and output function (40), to deduce the following result

Proposition 4 *Suppose the following hypothesis to hold*

i) $\hat{C}_i$ *commutes with* A_j *for all i and j such that* $0 \leqslant i \leqslant s, \ 0 \leqslant j \leqslant r$.

ii) $\|\sum_{i=0}^{r} A_i z_i\| \leqslant \varepsilon$ *for all* $(z_0, \dots, z_r) \in \underbrace{\hat{\mathcal{B}}_n(0,\varepsilon) \times \dots \times \hat{\mathcal{B}}_n(0,\varepsilon)}_{(r+1)-times}$

(where $\hat{\mathcal{B}}_n(0,\varepsilon) = \mathcal{B}_p(0,\varepsilon) \times 0_{\mathbb{R}^{n-p}} \subset \mathbb{R}^n$*)*

then $W_d(\varepsilon)$ *is finitely accessible, moreover* $W_d(\varepsilon) = W_d^r(\varepsilon)$.

6. Conclusion

In this paper the problem of the characterization of the admissible disturbances set for perturbed nonlinear discrete systems is considered. An efficient algorithm for constructing the admissible set is given and numerical simulations have been done for some examples. The case of controlled discrete-time delayed systems has also been investigated. As a natural continuation of this work its interest to investigate the same problem where the dynamics of the system is also perturbed, that means, to characterize the ε-admissibles perturbations from the system

$$\begin{cases} x^e(i+1) = (A+\delta)x^e(i) + f(\zeta_i u_i + \omega_i), \quad i \geqslant 0 \\ x^e(0) = \gamma x_0 + \psi \end{cases}$$

where δ is a perturbation which enters the dynamic of system.

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