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Computing anisotropic norm of linear discrete-time-invariant system via LMI-based approach

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The anisotropic norm of a linear discrete-time-invariant system is a measure of system output sensitivity to stationary Gaussian input disturbances with mean anisotropy bounded by some nonnegative parameter. The mean anisotropy characterizes the predictability (or coloredness) degree of stochastic signal. The anisotropic norm of a system is an induced norm, which limiting cases are $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms as $\alpha \to 0$ and $\alpha \to \infty$, respectively. In [1] a method for numerical computation of the anisotropic norm was proposed. This method involves linked Riccati and Lyapunov equations and associated special type equation. This paper develops a method for computing the anisotropic norm that reduces to finding a strongly rank-minimizing solution of linear matrix inequality and a solution of special type nonlinear algebraic equation.

Key words: differential entropy, anisotropy, induced norm, linear matrix inequality

Dedicated to the blessed memory of my dear mama.
Michael Tchaikovsky

1. Introduction

The well known $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimization approaches for linear time-invariant control systems are based on using $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms in the respective Hardy spaces of matrix-valued transfer functions. The performance criteria applied in these approaches are determined by different assumptions about the nature of input disturbances affecting the system.

The principal assumption of $\mathcal{H}_2$-theory posits that input disturbance is Gaussian white noise. $\mathcal{H}_\infty$-theory treats the plant modelling errors as a norm-bounded perturbation of a nominal transfer function and input disturbances as square-summable signals.

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A stochastic approach to $\mathcal{H}_\infty$-optimization of discrete-time-invariant control systems is based on using a stochastic norm in performance criteria. The stochastic norm measures the sensitivity of the system output to random input disturbances whose probability distribution is not precisely known.

According to [1, 2], the α-anisotropic norm of a system is a particular case of the stochastic norm and defined as the supremum of the ratio of the root mean square value of the system output to that of the input over all stationary Gaussian inputs with the mean anisotropy upper-bounded by a nonnegative parameter α.

For an absolutely continuously distributed Gaussian random vector, anisotropy is defined as difference between the differential entropy of a Gaussian random vector with zero mean and constant diagonal covariance matrix and the differential entropy of this vector, and can be considered as a measure of distinction between the covariance matrix of a random vector and the identity matrix [3]. This quantity is always nonnegative. Considering finite, but arbitrarily long initial segments of a stationary Gaussian sequence as such vectors, one can define the anisotropy per unit of time and the mean anisotropy as its limiting value. The mean anisotropy is an entropy-based measure of predictability (or coloredness) of Gaussian signal.

For any fixed mean anisotropy level $\alpha > 0$, the α-anisotropic norm lies between the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms and induces an intermediate topology, weaker than the $\mathcal{H}_\infty$-topology and stronger than the respective $\mathcal{H}_2$-topology. Moreover, the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of a system are limiting cases of the α-anisotropic norm as $\alpha \to 0$ and $\alpha \to +\infty$, correspondingly. The 0-anisotropic norm coincides with the $\mathcal{H}_2$-norm up to a constant multiplier depending on the dimension of the system input.

This paper presents approaches to computing the mean anisotropy of Gaussian sequence and anisotropic norm of linear discrete-time-invariant system based on solving the semidefinite programming (SDP) problem, which is to find strongly rank-minimizing solutions of corresponding linear matrix inequalities [4].

In this paper, the symbol $\doteq$ is used for definitions, and the symbol $\triangleq$ is used for denotations.

2. Backgrounds

2.1. Anisotropy of random vector

Let $\mathbb{L}_2^m$ denote the space of random vectors $W \in \mathbb{R}^m$ with absolutely continuous distribution and bounded second moment $\mathbf{E}|W|^2 < +\infty$, where $\mathbf{E}(\cdot)$ denotes the expectation and $|\cdot|$ is the Eucledian vector norm. Let vector W have the distribution density function $f(\cdot)$. Let $\mathbb{G}^m(\mu, \Sigma)$ be the space of m-dimensional random vectors W such that $\mathbf{E}(W) = \mu$ and the covariance matrix $\mathbf{cov}(W) = \mathbf{E}\left((W-\mu)(W-\mu)^{\mathrm{T}}\right) = \Sigma$.

The anisotropy of a random vector $W \in \mathbb{L}_2^m$ is defined as follows [3]

$$\mathbf{A}(W) = \mathbf{h}(W_g) - \mathbf{h}(W), \tag{1}$$

where $W_g \in \mathbb{G}\left(0, m^{-1}\mathbf{E}|W|^2 I_m\right)$ and $\mathbf{h}(W)$ is the differential entropy of vector W defined as [5, 6]

$$\mathbf{h}(W) = -\mathbf{E}\left(\log f(W)\right) = -\int\limits_{\mathbb{R}^m} f(w)\log f(w)\, dw.$$

The value

$$\mathbf{h}(W_g) = \frac{m}{2}\log\left(\frac{2\pi \mathrm{e}}{m}\mathbf{E}|W|^2\right)$$

is the differential entropy of a random vector from the space $\mathbb{G}(0, m^{-1}\mathbf{E}|W|^2 I_m)$ with normal distribution, zero mean and covariance matrix $\mathbf{E}|W|^2 I_m$.

As one can see from definition, the anisotropy is in a sense a measure of distinction between the covariance matrix of a random vector and the identity matrix. Following this, the anisotropy $\mathbf{A}(W)$ characterizes (directional) irregularity of distribution for the random vector W or, equivalently, uninvariance of this vector with respect to unitary transformation. Such understanding of the value $\mathbf{A}(W)$ justifies the term *anisotropy*. The notion of anisotropy leads to new statements of stochastic $\mathcal{H}_\infty$-optimization problems in presence of Gaussian random disturbances and gives substantial physical interpretation of informational criteria arising in stochastic control problems [7].

Let us formulate the major properties of the anisotropy functional.

Lemma 1 [3]

1. *The anisotropy* $\mathbf{A}(W)$ *is invariant under rotation and central dilatation, that is* $\mathbf{A}(\lambda U W) = \mathbf{A}(W)$ *for any unitary matrix* U *and any scalar value* $\lambda \neq 0$*;*

2. *For any positive definite matrix* $\Sigma \in \mathbb{R}^{m\times m}$,

$$\min\left\{\mathbf{A}(W) : W \in \mathbb{L}_2^m \wedge \mathbf{E}(WW^{\mathrm{T}}) = \Sigma\right\} = -\frac{1}{2}\log\det\left(\frac{m\Sigma}{\operatorname{tr}\Sigma}\right), \tag{2}$$

   and the minimum is reached only at $W \in \mathbb{G}^m(0, \Sigma)$*;*

3. $\mathbf{A}(W) \geqslant 0$ *and* $\mathbf{A}(W) = 0$ *if and only if* $W \in \mathbb{G}^m(0, \lambda I_m)$ *for some* $\lambda > 0$.

For subsequent reasoning we need the following result. Let an arbitrary vector $W \in \mathbb{L}_2^m$ be partitioned into the blocks $w_1, w_2, \ldots, w_r$ of dimensions $m_1, m_2, \ldots, m_r$, correspondingly, and $m_1 + m_2 + \ldots + m_r = m$, that is

$$W = \begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix}. \tag{3}$$

For any $1 \leqslant s \leqslant t \leqslant r$, we will use the notation $W_{s:t} = (w_k)_{s\leqslant k\leqslant t}$ for subvector of vector (3) consisting of elements $w_s, w_{s+1}, \ldots, w_t$.

Let $\mathbf{I}(w_1;w_2)$ denote the mutual information contained in the vector $w_2 \in \mathbb{R}^{m_2}$ with respect to the vector $w_1 \in \mathbb{R}^{m_1}$ defined as [5, 6]:

$$\mathbf{I}(w_1;w_2) = \iint\limits_{\mathbb{R}^{m_1}\mathbb{R}^{m_2}} f(w_1,w_2)\log\frac{f(w_1,w_2)}{f(w_1)f(w_2)}\,dw_1dw_2, \tag{4}$$

where $f(w_1,w_2)$ is the joint distribution density function, and $f(w_1)$, $f(w_2)$ are the distribution density functions of w_1 and w_2, respectively. The mutual information $\mathbf{I}(w_1;w_2)$ can be expressed via differential entropy as follows:

$$\mathbf{I}(w_1;w_2) = \mathbf{h}(w_1) - \mathbf{h}(w_1|w_2),$$

where $\mathbf{h}(w_1|w_2)$ is the conditional differential entropy defined as

$$\mathbf{h}(w_1|w_2) = -\iint\limits_{\mathbb{R}^{m_1}\mathbb{R}^{m_2}} f(w_1,w_2)\log f(w_1|w_2)\,dw_1dw_2$$

and the conditional distribution function $f(w_1|w_2) = f(w_1,w_2)/f(w_2)$.

Lemma 2 [3] *Anisotropy (1) of random vector (3) is representable as*

$$\mathbf{A}(W) = \mathbf{A}(U) + \sum_{k=1}^{r}\mathbf{A}(w_k) + \sum_{k=2}^{r}\mathbf{I}(w_k;W_{1:k-1}),$$

where $U \in \mathbb{G}^m(0,\Sigma)$ has the block-diagonal covariance matrix

$$\Sigma = \mathrm{blockdiag}\left(\frac{1}{m_k}\mathbf{E}|w_k|^2 I_{m_k}\right).$$

2.2. Mean anisotropy of Gaussian signal

Let $\mathcal{P}^m$ be the space of square-summable vector sequences $(w_k)_{-\infty<k<\infty}$, where $w_k \in \mathbb{L}_2^m$, that is

$$\mathcal{P}^m = \{W = (w_k)_{-\infty<k<\infty} : w_k \in \mathbb{L}_2^m \wedge \|W\|_{\mathcal{P}} < +\infty\},$$

where $\|\cdot\|_{\mathcal{P}}$ is the power semi-norm of sequence defined as [8]

$$\|W\|_{\mathcal{P}} \doteq \left\{\lim_{N\to\infty}\frac{1}{2N+1}\sum_{k=-N}^{N}\mathbf{E}|w_k|^2\right\}^{1/2} = \sqrt{\mathrm{tr}\,R_{ww}(0)}.$$

Here $R_{ww}(n)$ denotes the covariance function of the sequence $(w_k)_{-\infty<k<\infty}$. The power semi-norm of sequence can be determined via its spectral density S_{ww} as follows [8]

$$\|W\|_{\mathcal{P}} = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}\,\widehat{S}_{ww}(\omega)d\omega\right\}^{1/2},$$

where

$$\widehat{S}_{ww}(\omega) \triangleq \lim_{r\to 1-0} S_{ww}\left(r\,\mathrm{e}^{i\omega}\right), \quad -\pi \leqslant \omega < \pi,$$

and $r \to 1-0$ means "r tends to one from the left."

Let $V = (v_k)_{-\infty<k<+\infty}$ be m-dimensional discrete-time Gaussian white noise with zero mean and identity covariance matrix [1]:

$$\mathbf{E}(v_k) = 0, \quad \mathbf{E}(v_k v_k^{\mathrm{T}}) = \delta_{jk} I_m, \quad -\infty < k < +\infty,$$

where δ_{jk} is the Kronecker delta. Consider the m-dimensional stationary Gaussian sequence

$$W = (w_k)_{-\infty<k<+\infty} = G * V,$$

generated from the white noise V by a stable generating filter G with an impulse response function $g_k \in \mathbb{R}^{m\times m}, k \geqslant 0$:

$$w_j = \sum_{k=0}^{+\infty} g_k v_{j-k}, \quad -\infty < j < +\infty. \tag{5}$$

This generating filter is identified with its transfer matrix

$$G(z) = \sum_{k=0}^{+\infty} g_k z^k, \tag{6}$$

which is assumed to be in the Hardy space $\mathcal{H}_2^{m\times m}$ with $\mathcal{H}_2$-norm

$$\|G\|_2 \doteq \left\{\sum_{k=0}^{+\infty} \mathrm{tr}\left(g_k g_k^{\mathrm{T}}\right)\right\}^{1/2} = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}\left(\widehat{G}(\omega)\widehat{G}^*(\omega)\right)d\omega\right\}^{1/2}, \tag{7}$$

where $G^*(z) \triangleq G^{\mathrm{T}}(\overline{z})$.

The series in the right part of (5) converges in mean square sense. The sequence W has zero mean and spectral density

$$\widehat{S}_{ww}(\omega) = \widehat{G}(\omega)\widehat{G}^*(\omega), \quad \omega \in [-\pi, \pi]. \tag{8}$$

Now, let us extend the vector anisotropy notion onto the case of Gaussian stationary random sequence $W = (w_k)_{-\infty<k<\infty}$, where $w_k \in \mathbb{G}^m(0,\Sigma)$.

Let $W = (w_k)_{-\infty<k<+\infty}$ be an ergodic strictly stationary sequence of m-dimensional random vectors with $\mathbf{E}|w_k|^2 < +\infty$, and all the finite-dimensional probability distributions of this sequence be absolutely continuous. Define the mean anisotropy of the sequence W as follows

$$\overline{\mathbf{A}}(W) \doteq \lim_{N\to+\infty} \frac{\mathbf{A}(W_{0:N-1})}{N}.$$

[1] Here and below, the subscripts of sequence elements are the integers, and for brevity only the range of their variation is indicated.

Let us show that the limit in the above definition of the mean anisotropy always exists. Applying Lemma 2, we have

$$\frac{1}{N}\mathbf{A}(W_{0:N-1}) = \mathbf{A}(w_0) + \frac{1}{N}\sum_{k=1}^{N-1}\mathbf{I}(w_0; W_{-k:-1}),$$

whence it follows that

$$\overline{\mathbf{A}}(W) = \lim_{N\to+\infty}\frac{\mathbf{A}(W_{0:N-1})}{N} = \mathbf{A}(w_0) + \mathbf{I}(w_0; (w_k)_{k<0}). \tag{9}$$

Here $\mathbf{I}(w_0; (w_k)_{k<0}) = \lim_{k\to+\infty}\mathbf{I}(w_0; W_{-k:-1})$ represents the mean mutual information between the element w_0 of the sequence $(w_k)_{-\infty<k<\infty}$ and the past history $(w_k)_{k<0}$. For Gaussian random sequences, this quantity can be determined by the formula

$$\mathbf{I}(w_0; (w_k)_{k<0}) = \frac{1}{2}\log\det(\mathbf{E}(w_0 w_0^{\mathrm{T}})(\mathbf{E}(\widetilde{w}_0\widetilde{w}_0^{\mathrm{T}}))^{-1}), \tag{10}$$

where $\widetilde{w}_0 = w_0 - \mathbf{E}(w_0|(w_k)_{k<0})$ is the error of best (in means-quare sense) value prediction for the element w_0 of the sequence $(w_k)_{-\infty<k<\infty}$ via the known realization of its past history $(w_k)_{k<0}$ [3].

Using (2) and (10), one can rewrite expression (9) in the following form

$$\overline{\mathbf{A}}(W) = -\frac{1}{2}\log\det\frac{m\mathbf{E}(\widetilde{w}_0\widetilde{w}_0^{\mathrm{T}})}{\mathbf{E}|w_0|^2}. \tag{11}$$

According to Kolmogorov-Szegö formula [9],

$$\log\det(\mathbf{E}(\widetilde{w}_0\widetilde{w}_0^{\mathrm{T}})) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\det\widehat{S}_{ww}(\omega)d\omega. \tag{12}$$

Comparison of formulas (11) and (12) taking into account (8) yields

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\log\det\left\{\frac{m}{\|G\|_2^2}\widehat{G}(\omega)\widehat{G}^*(\omega)\right\}d\omega. \tag{13}$$

Expression (13) is the formula for the mean anisotropy of Gaussian stationary random sequence. The sequence W is completely defined by its generating filter G, therefore, together with the notation $\overline{\mathbf{A}}(W)$, we will use the equivalent notation $\overline{\mathbf{A}}(G)$.

Functional (13) has a nonnegative finite value, if the generating filter $G \in \mathcal{H}_2^{m\times m}$ has the full row rank, that is

$$\operatorname{rank}\widehat{G}(\omega) = m \quad \text{for almost all} \quad \omega \in [-\pi, \pi).$$

If the filter G does not have maximum rank, then $\overline{\mathbf{A}}(G) = +\infty$. The equality $\overline{\mathbf{A}}(G) = 0$ holds true if and only if the sequence W is the Gaussian white noise with the diagonal constant covariance matrix. It means that there exists some number $\lambda > 0$ such that the following equality holds true:

$$G(\omega)G^*(\omega) = \lambda I_m, \quad -\pi \leqslant \omega < \pi. \tag{14}$$

2.3. State-space formulas for mean anisotropy

Let the generating filter $G \in \mathcal{H}_2^{m\times m}$ has the following state-space realization:

$$\left.\begin{aligned} x_{k+1} &= Ax_k + Bv_k \\ w_k &= Cx_k + Dv_k \end{aligned}\right\}, \quad -\infty < k < +\infty, \tag{15}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$. Suppose that the matrix A is asymptotically stable (its spectral radius $\rho(A) < 1$), and the matrix D is nonsingular. For the transfer matrix $G(z)$, the following equality holds:

$$G(z) = C(zI - A)^{-1}B + D.$$

The transfer matrix $G(z)$ is said to have state-space (A,B,C,D)-realization, and this fact is denoted by $G(z) = (A,B,C,D)$ or

$$G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

In [1], the following theorem presenting the formulas for computing the mean anisotropy of the sequence W generated by the filter with realization (15) was formulated and proved.

Theorem 1 *Let a generating filter $G \in \mathcal{H}_2^{m\times m}$ have state-space realization (15) with A asymptotically stable and D nonsingular. Then mean anisotropy (13) of the sequence $W = G * V$ is*

$$\overline{\mathbf{A}}(G) = -\frac{1}{2}\log\det\left\{\frac{m(CRC^{\mathrm{T}} + DD^{\mathrm{T}})}{\mathrm{tr}(CPC^{\mathrm{T}} + DD^{\mathrm{T}})}\right\}, \tag{16}$$

where the matrix $R \in \mathbb{R}^{n\times n}$ is the admissible solution of algebraic Riccati equation

$$ARA^{\mathrm{T}} - R - (ARC^{\mathrm{T}} + BD^{\mathrm{T}})(CRC^{\mathrm{T}} + DD^{\mathrm{T}})^{-1}(CRA^{\mathrm{T}} + DB^{\mathrm{T}}) + BB^{\mathrm{T}} = 0, \tag{17}$$

and the matrix P is the controllability gramian of the generating filter satisfying Lyapunov equation

$$APA^{\mathrm{T}} - P + BB^{\mathrm{T}} = 0. \tag{18}$$

Remark 1 *Remind that the matrix $R \in \mathbb{R}^{n\times n}$ is said to be admissible solution of algebraic Riccati equation (17) if it satisfies this Riccati equation, is symmetric, positive semidefinite, and such that the matrix*

$$A - (ARC^{\mathrm{T}} + BD^{\mathrm{T}})(CRC^{\mathrm{T}} + DD^{\mathrm{T}})^{-1}C$$

is asymptotically stable, that is its spectral radius

$$\rho\left\{A - (ARC^{\mathrm{T}} + BD^{\mathrm{T}})(CRC^{\mathrm{T}} + DD^{\mathrm{T}})^{-1}C\right\} < 1.$$

Remark 2 *It should be noted that for generating filter (15) with asymptotically stable matrix A and transfer matrix $G \in \mathcal{H}_2^{m\times m}$ the following identities hold true:*

$$\mathbf{E}(\widetilde{w}_0\widetilde{w}_0^{\mathrm{T}}) \equiv CRC^{\mathrm{T}} + DD^{\mathrm{T}}, \tag{19}$$

$$\mathbf{E}|w_0|^2 \equiv \mathrm{tr}\{CPC^{\mathrm{T}} + DD^{\mathrm{T}}\}. \tag{20}$$

2.4. Anisotropic norm of linear system

Let F be physically realizable linear discrete-time-invariant system with m-dimensional input W and p-dimensional output $Z = F * W$. The realization of the random sequence feeding the input W is called to be the signal and identified with the system input. Assume that the transfer matrix of the system F belongs to the Hardy space $\mathcal{H}_\infty^{p\times m}$, that is F is analytical in the open unit disc of the complex plane and has finite $\mathcal{H}_\infty$-norm

$$\|F\|_\infty \doteq \sup_{|z|<1} \overline{\sigma}(F(z)) = \operatorname*{ess\,sup}_{-\pi\leqslant\omega<\pi} \overline{\sigma}\left(\widehat{F}(\omega)\right), \tag{21}$$

where $\overline{\sigma}(\cdot)$ denotes the maximum singular value of the matrix F.

Let us introduce the set

$$\mathcal{G}_\alpha \triangleq \left\{G \in \mathcal{H}_2^{m\times m} :\ \overline{\mathbf{A}}(G) \leqslant \alpha\right\} \tag{22}$$

of generating filters producing the Gaussian sequences $W = G * V$ with the mean anisotropy upper-bounded by some given nonnegative parameter α. The respective family of spectral densities (8) is given by $\mathbf{S}_\alpha = \{\widehat{G}\widehat{G}^* :\ G \in \mathcal{G}_\alpha\}$.

α-Anisotropic norm of the system $F \in \mathcal{H}_\infty^{p\times m}$ is defined as

$$|||F|||_\alpha \doteq \sup\left\{\frac{\|FG\|_2}{\|G\|_2} :\ G \in \mathcal{G}_\alpha\right\}, \quad \alpha \geqslant 0, \tag{23}$$

and is the special case of the more general stochastic system norm in respect of the family of input probability distributions [10, 7]. In terms of input-output signals, the anisotropic norm of the system F can be presented as follows

$$|||F|||_a = \sup_{G\in\mathcal{G}_a} \frac{\|FG\|_2}{\|G\|_2} = \sup_{W\in\mathcal{W}_a} \frac{\|Z\|_{\mathcal{P}}}{\|W\|_{\mathcal{P}}} = \sup_{W\in\mathcal{B}\mathcal{W}_a} \|Z\|_{\mathcal{P}}, \tag{24}$$

where $\mathcal{W}_a$ is the set of input signals with bounded mean anisotropy value:

$$\mathcal{W}_\alpha = \{W \in \mathcal{P}^m : W = G * V,\ \text{where } V \in \mathbb{G}^m(0, I_m), G \in \mathcal{G}_\alpha\},$$

and, similarly, $\mathcal{B}\mathcal{W}_\alpha$ is the set of normalized input signals with bounded mean anisotropy value:

$$\mathcal{B}\mathcal{W}_\alpha = \left\{W \in \mathcal{W}_\alpha : \|W\|_{\mathcal{P}} = 1\right\}.$$

For any fixed system $F \in \mathcal{H}_\infty^{p\times m}$, its α-anisotropic norm (23) is a nondecreasing continuous function of parameter $\alpha \geqslant 0$ and satisfies inequality

$$\frac{1}{\sqrt{m}}\|F\|_2 = |||F|||_0 \leqslant \lim_{\alpha\to+\infty} |||F|||_\alpha = \|F\|_\infty, \tag{25}$$

which indicates that the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms are the limiting cases of the anisotropic norm.

Computing the norm $|||F|||_\alpha$ is only of interest in that cases when $\alpha > 0$ and the inequality

$$\frac{1}{\sqrt{m}}\|F\|_2 < \|F\|_\infty \tag{26}$$

holds true. Otherwise, the anisotropic norm trivially coincides with $\mathcal{H}_2$-norm up to the constant multiplier $\frac{1}{\sqrt{m}}$. It can be easily shown that condition (26) does not hold only for system satisfying (14).

2.5. Frequency-domain formulas for anisotropic norm

Associate with anisotropic norm (23) the following set of *worst-case input generating filters*

$$\mathcal{G}_\alpha^\diamond \triangleq \left\{ G \in \mathcal{G}_\alpha : \frac{\|FG\|_2}{\|G\|_2} \to \max \right\}. \tag{27}$$

In [1] it was shown that the set $\mathcal{G}_\alpha^\diamond$ can be characterized as follows. Define the functions

$$\mathcal{A}(q) = \frac{m}{2}\{\log\Phi(q) - \Psi(q)\}, \tag{28}$$

$$\mathcal{N}(q) = \left\{\frac{1}{q}\left(1 - \frac{1}{\Phi(q)}\right)\right\}^{1/2}, \tag{29}$$

where

$$\Phi(q) = \frac{1}{2\pi m}\int_{-\pi}^{\pi} \operatorname{tr}(I_m - q\Lambda(\omega))^{-1}\, d\omega, \tag{30}$$

$$\Psi(q) = -\frac{1}{2\pi m}\int_{-\pi}^{\pi} \log\det(I_m - q\Lambda(\omega))\, d\omega \tag{31}$$

and

$$\Lambda(\omega) = \widehat{F}^*(\omega)\widehat{F}(\omega), \quad \omega \in [-\pi, \pi]. \tag{32}$$

Let us note that functions (28)–(31) are analytic for $\forall q \in \left[0, \gamma^{-2}\right)$, where $\gamma = \|F\|_\infty$. Furthermore, for any generating filter $G \in \mathcal{H}_\infty^{m\times m}$ satisfying the equality

$$\widehat{G}(\omega)\widehat{G}^*(\omega) = (I_m - q\Lambda(\omega))^{-1}, \quad \omega \in [-\pi, \pi), \tag{33}$$

the following equalities hold true:

$$\overline{\mathbf{A}}(G) = \mathcal{A}(q), \quad \frac{\|FG\|_2}{\|G\|_2} = \mathcal{N}(q), \quad \|G\|_2^2 = m\Phi(q). \tag{34}$$

Factorization (33) exists for any generating filter $G \in \mathcal{H}_\infty^{m\times m}$ owing to condition $q < \gamma^{-2}$.

In [1] the following theorem was formulated and proved establishing the frequency-domain formula for computing the α-anisotropic norm of the system with transfer matrix $F(z)$.

Theorem 2 *Let the system with transfer function $F \in \mathcal{H}_\infty^{p\times m}$ satisfy strict inequality (26). Then:*

1. *function (28) is analytic, strictly increasing and convex on $q \in [0, \gamma^{-2})$, and the concave inverse function $\mathcal{A}^{-1} : \mathbb{R}_+ \to [0, \gamma^{-2})$ exists;*
2. *α-anisotropic norm (23) of the system with transfer matrix $F(z)$ is expressed as*

$$|||F|||_\alpha = \mathcal{N}\left(\mathcal{A}^{-1}(\alpha)\right); \tag{35}$$

3. *any filter $G_q \in \mathcal{H}_\infty^{m\times m}$ capable to factorization (33) with the parameter $q = \mathcal{A}^{-1}(\alpha)$ belongs to set (27) of the worst-case generating filters, in other words, $G_q \in \mathcal{G}_\alpha^\diamond$.*

Thus, computing the α-anisotropic norm by formula (35) leads to solving the equation

$$\mathcal{A}(q) = \alpha \tag{36}$$

on the parameter q for given α. Since the function $\mathcal{A}(q)$ is monotonic, convex and smooth, it is natural to use the Newton's method for solving nonlinear algebraic equation (36).

2.6. State-space formulas for anisotropic norm

Let the system with transfer matrix $F \in \mathcal{H}_\infty^{p\times m}$, n-dimensional internal state X, m-dimensional input W and p-dimensional output $Z = G * W$ be described by the following equations:

$$\left.\begin{aligned} x_{k+1} &= Ax_k + Bw_k \\ z_k &= Cx_k + Dw_k \end{aligned}\right\}, \quad -\infty < k < +\infty, \tag{37}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$, and the matrix A is asymptotically stable. Thus, this system with transfer matrix $F(z)$ is presented by the realization

$$F(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

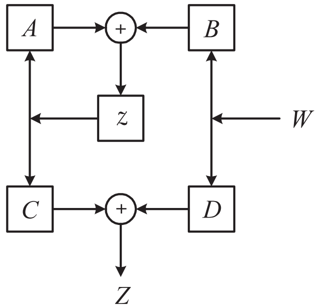

Figure 1. Block-diagram of system with transfer matrix $G(z) = (A,B,C,D)$

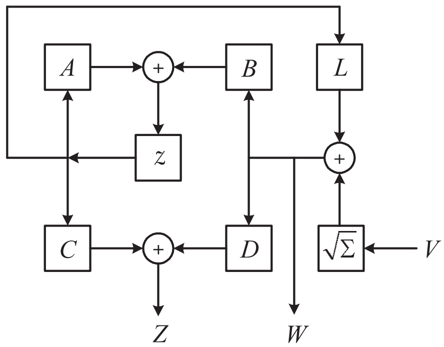

Figure 2. Block-diagram of generating filter (38)

Block-diagram of the system with transfer matrix $F(z) = (A,B,C,D)$ is presented at Figure 1.

Let $L \in \mathbb{R}^{m\times n}$ be a matrix such that $A+BL$ is asymptotically stable, and let $\Sigma \in \mathbb{R}^{m\times m}$ be a positive definite symmetric matrix. Consider the generating filter with transfer matrix $G(z),$ input V and output W presented by the realization

$$G(z) = \left[\begin{array}{c|c} A+BL & B\Sigma^{1/2} \\ \hline L & \Sigma^{1/2} \end{array}\right].$$

The equations for this generating filter are given by

$$\left.\begin{array}{ll} x_{k+1} & = (A+BL)x_k + B\Sigma^{1/2}v_k \\ w_k & = Lx_k + \Sigma^{1/2}v_k \end{array}\right\}, \quad -\infty < k < +\infty. \tag{38}$$

Block-diagram of generating filter (38) is presented at Figure 2.

The following theorem was formulated and proved in [2].

Theorem 3 *Let system (37) with asymptotically stable matrix A satisfying strict inequality (26) and the mean anisotropy level α of the input sequence W be given. Then there exists a unique pair (q,R) of the parameter $q \in [0,\gamma^{-2})$ and the matrix $R \in \mathbb{R}^{n\times n}$, which is an admissible solution of algebraic Riccati equation*

$$A^{\mathrm{T}}RA - R - (A^{\mathrm{T}}RB + qC^{\mathrm{T}}D)(B^{\mathrm{T}}RB + qD^{\mathrm{T}}D - I_m)^{-1} \times (B^{\mathrm{T}}RA + qD^{\mathrm{T}}C) + qC^{\mathrm{T}}C = 0 \quad (39)$$

such that

$$-\frac{1}{2}\log\det\left\{\frac{m\Sigma}{\mathrm{tr}\,(LPL^{\mathrm{T}}+\Sigma)}\right\} = \alpha \quad (40)$$

as

$$\Sigma \triangleq (I_m - qD^{\mathrm{T}}D - B^{\mathrm{T}}RB)^{-1}, \quad (41)$$

$$L \triangleq \Sigma(B^{\mathrm{T}}RA + qD^{\mathrm{T}}C), \quad (42)$$

where the matrix $P \in \mathbb{R}^{n\times n}$ is the controllability gramian of generating filter

$$G(z) = \left[\begin{array}{c|c} A+BL & B\Sigma^{1/2} \\ \hline L & \Sigma^{1/2}\end{array}\right] \quad (43)$$

satisfying Lyapunov equation

$$(A+BL)P(A+BL)^{\mathrm{T}} - P + B\Sigma B^{\mathrm{T}} = 0. \quad (44)$$

At that, filter (43) is the worst-case generating filter, and the α-anisotropic norm of the system F is defined to be

$$|||F|||_\alpha = \left\{\frac{1}{q}\left(1 - \frac{m}{\mathrm{tr}\,(LPL^{\mathrm{T}}+\Sigma)}\right)\right\}^{1/2}. \quad (45)$$

Remark 3 *It should be noted that for $\forall q \in [0,\gamma^{-2})$ Riccati equation (39) has a unique admissible solution $R \geqslant 0$.*

Remark 4 *In [1] it was shown that for asymptotically stable system (37) functions (28)–(31) are explicitly given by*

$$\mathcal{A}(q) = -\frac{1}{2}\log\det\left\{\frac{m\Sigma}{\mathrm{tr}\,(LPL^{\mathrm{T}}+\Sigma)}\right\}, \quad (46)$$

$$\mathcal{N}(q) = \left\{\frac{1}{q}\left(1 - \frac{m}{\mathrm{tr}\,(LPL^{\mathrm{T}}+\Sigma)}\right)\right\}^{1/2}, \quad (47)$$

$$\Phi(q) = \frac{1}{m}\mathrm{tr}\,(LPL^{\mathrm{T}}+\Sigma), \quad (48)$$

$$\Psi(q) \quad = \quad \frac{1}{m}\log\det\Sigma, \tag{49}$$

where matrices Σ and L connect to the admissible solution R of algebraic Riccati equation (39) via expressions (41), (42), and the matrix P is the controllability gramian of worst-case generating filter (43) satisfying Lyapunov equation (44).

2.7. Newton's method for computing anisotropic norm

Consider one of the methods for computing the anisotropic norm of assymptotically stable system (37) with transfer matrix

$$F(z) = (A,B,C,D)$$

presented in [1].

For given level of mean anisotropy $\alpha > 0$, computing the α-anisotropic norm of asymptotically stable system (37) satisfying inequality (26) leads to solving the nonlinear algebraic equation

$$\mathcal{A}(q) = \alpha \tag{50}$$

on the parameter $q \in \left[0,\gamma^{-2}\right)$. One can naturally use Newton's method to find the solution of equation (50). More precisely, define the sequence of real numbers $(q_k)_{k\geqslant 0}$ recursively by

$$q_{k+1} = \begin{cases} (\gamma^{-2}+q_k)/2 & \text{at} \quad \mathcal{A}(q_k) < \alpha, \\ q_k + (\alpha - \mathcal{A}(q_k))/\mathcal{A}'(q_k) & \text{at} \quad \mathcal{A}(q_k) \geqslant \alpha, \end{cases} \qquad q_0 = 0. \tag{51}$$

By smoothness, monotonicity and convexity of the function $\mathcal{A}(q)$, the sequence (q_k) converges to the limit

$$\lim_{k\to+\infty} q_k = \mathcal{A}^{-1}(\alpha).$$

This convergence is eventually monotonic in the sense that

$$\mathcal{A}^{-1}(\alpha) \leqslant q_{k+1} \leqslant q_k \quad \text{for} \quad \forall k \geqslant k_*,$$

where

$$k_* = \min_{k\geqslant 1}\left\{\mathcal{A}\left((1-2^k)\gamma^{-2}\right) \geqslant \alpha\right\}.$$

In practical computations, the condition

$$(\mathcal{A}(q_k)-\alpha)\max\left\{1, \frac{\mathcal{N}'(q_k)}{\mathcal{A}'(q_k)\mathcal{N}(q_k)}\right\} < \varepsilon \tag{52}$$

can be used to stop the iterative procedure, and this provides the relative accuracy ε for computing the α-anisotropic norm of the system. The derivatives of functions $\mathcal{A}(q)$ and

$\mathcal{N}(q)$ given by expressions (46) and (47), respectively, and participating in (51) and (52), are defined to be

$$\mathcal{A}'(q) = \frac{1}{2q}\left\{m\frac{\|G\|_4^4}{\|G\|_2^2} - \|G\|_2^2\right\}, \tag{53}$$

$$\mathcal{N}'(q) = \frac{1}{2}\mathcal{N}(q)\left\{\frac{\Phi'(q)}{\Phi(q)(\Phi(q)-1)} - \frac{1}{q}\right\}, \tag{54}$$

where $\Phi(q)$ is determined by (48) and

$$\Phi'(q) = \frac{1}{mq_k}\left(\|G\|_4^4 - \|G\|_2^2\right). \tag{55}$$

Formulas (53)–(55) use the $\mathcal{H}_4$-norm of worst-case generating filter (43) that can be expressed in terms of the controllability gramian $P \in \mathbb{R}^{n\times n}$ satisfying Lyapunov equation (44) as well as the observability gramian $Q \in \mathbb{R}^{n\times n}$ satisfying the Lyapunov equation

$$(A+BL)^{\mathrm{T}}Q(A+BL) - Q + L^{\mathrm{T}}L = 0 \tag{56}$$

as follows:

$$\|G\|_4^4 = \mathrm{tr}\{LPL^{\mathrm{T}} + \Sigma\}^2 + 2\,\mathrm{tr}\{K^{\mathrm{T}}QK\}, \tag{57}$$

where

$$K = (A+BL)PL^{\mathrm{T}} + B\Sigma,$$

and the matrices Σ and L are given by (41) and (42), respectively.

Subject to equalities (34) and (48), the square of $\mathcal{H}_2$-norm of the transfer matrix $G(z)$ of the worst-case generating filter is defined to be

$$\|G\|_2^2 = \mathrm{tr}\left\{LPL^{\mathrm{T}} + \Sigma\right\}. \tag{58}$$

The following algorithm summarizes aforesaid approach to computing the α-anisotropic norm $|||F|||_\alpha$ of system (37) using Newton's method for solving the nonlinear algebraic equation.

Algorithm 1 *Give the relative accuracy ε for computing the α-anisotropic norm. Find $\|F\|_\infty = \gamma$. Set the initial value of the parameter $q_0 = 0$ and the step number $k = 0$.*

1. *For current value of the parameter q_k, find the admissible solution R of algebraic Riccati equation (39), the solution P of Lyapunov equation (44), and the solution Q of Lyapunov equation (56).*

2. *Find $\|G\|_4^4$ and $\|G\|_2^2$ using formulas (57), (58), correspondingly.*

3. *For current value of the parameter q_k, compute the values of functions $\mathcal{A}(q_k), \mathcal{N}(q_k)$, and $\Phi(q_k)$ using formulas (46), (47), and (48), as well as the values of their derivatives $\mathcal{A}'(q_k), \mathcal{N}'(q_k)$, and $\Phi'(q_k)$ by formulas (53), (54), and (55), respectively.*

4. *Check up whether condition (52) holds true. If the condition is satisfied, the computations stop, and, subject to expressions (45) and (47), the* α*-anisotropic norm* $|||F|||_\alpha = \mathcal{N}(q_k)$*. Otherwise, compute* q_{k+1} *using formula (51), set the step number* k *equal to* $k+1$*, and switch to point (1) of the algorithm.*

Thus, the computational algorithm for α-anisotropic norm $|||F|||_\alpha$ of system (37) is the iterative procedure with finding the admissible solution R of algebraic Riccati equation (39), the solution P of Lyapunov equation (44), and the solution Q of Lyapunov equation (56) on each step of iteration.

3. Computing mean anisotropy and anisotropic norm via LMI-based approach

3.1. Linear matrix inequalities and algebraic Riccati equations for discrete-time systems

Following [4], let us give some information relating to linear matrix inequalities as well as algebraic Riccati equations for the discrete-time systems.

Consider the linear matrix inequality

$$L(X) \triangleq \begin{bmatrix} A^{\mathrm{T}}XA - X + Q & A^{\mathrm{T}}XB + S^{\mathrm{T}} \\ B^{\mathrm{T}}XA + S & B^{\mathrm{T}}XB + R \end{bmatrix} \geqslant 0 \tag{59}$$

for an unknown matrix $X \in \mathbb{R}^{n\times n}$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $Q = Q^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $R = R^{\mathrm{T}} \in \mathbb{R}^{m\times m}$, and $S \in \mathbb{R}^{m\times n}$. Denote the set of real symmetric solutions of discrete linear matrix inequality (59) as

$$\Gamma \triangleq \left\{X \in \mathbb{R}^{n\times n} : X = X^{\mathrm{T}} \text{ and } L(X) \geqslant 0\right\}. \tag{60}$$

A solution X of linear matrix inequality (59) is said to be a *rank-minimizing solution* if

$$\operatorname{rank} L(X) = \beta \triangleq \min_{\forall Y \in \Gamma} \operatorname{rank} L(Y).$$

Denote the set of rank-minimizing solutions of discrete linear matrix inequality (59) as

$$\Gamma_{\min} \triangleq \{X \in \Gamma : \operatorname{rank} L(X) = \beta\}. \tag{61}$$

A solution $X \in \Gamma$ is said to be a *strongly rank-minimizing solution* if

$$\operatorname{rank} L(X) = \operatorname{rank}(B^{\mathrm{T}}XB + R). \tag{62}$$

Denote the set of strongly rank-minimizing solutions of linear matrix inequality (59) as

$$\mathcal{L}_{\min} \triangleq \{X \in \Gamma : \ \operatorname{rank} L(X) = \operatorname{rank}(B^{\mathrm{T}}XB + R)\}.$$

By Schur lemma (see, e.g., [11]), the matrix X satisfies inequality $L(X) \geqslant 0$ if and only if

$$A^{\mathrm{T}}XA - X - (A^{\mathrm{T}}XB + S^{\mathrm{T}})(B^{\mathrm{T}}XB + R)^{\dagger}(B^{\mathrm{T}}XA + S) + Q \geqslant 0, \tag{63}$$

$$B^{\mathrm{T}}XB + R \geqslant 0 \tag{64}$$

provided that

$$\ker(B^{\mathrm{T}}XB + R) \subseteq \ker(A^{\mathrm{T}}XB + S^{\mathrm{T}}).$$

In inequality (63), $M^{\dagger}$ denotes Moore-Penrose generalized inverse of the matrix M.

Consider the generalized discrete algebraic Riccati equation

$$\mathrm{Ric}(X) \triangleq A^{\mathrm{T}}XA - X - (A^{\mathrm{T}}XB + S^{\mathrm{T}})(B^{\mathrm{T}}XB + R)^{\dagger}(B^{\mathrm{T}}XA + S) + Q = 0, \tag{65}$$

$$\ker(B^{\mathrm{T}}XB + R) \subseteq \ker(A^{\mathrm{T}}XB + S^{\mathrm{T}}) \tag{66}$$

associated with linear matrix inequality (59) for an unknown matrix $X \in \mathbb{R}^{n\times n}$, where the given matrices are the same as in linear matrix inequality (59) provided that additional requirement

$$B^{\mathrm{T}}XB + R \geqslant 0 \tag{67}$$

holds true.

The following lemma formulated and proved in [4] establishes one-to-one correspondence between the set $\mathfrak{L}_{\min}$ and the set of real symmetric solutions of discrete algebraic Riccati equation (65), (66) with additional requirement (67).

Lemma 3 *The set of strongly rank-minimizing solutions of discrete linear matrix inequality (59) coincides with the set of real symmetric solutions of discrete algebraic Riccati equation associated with this linear matrix inequality. In other words, any real symmetric matrix X satisfying the generalized algebraic Riccati equation (65), (66) and additional condition (67) belongs to the set $\mathfrak{L}_{\min}$. Conversely, any matrix $X \in \mathfrak{L}_{\min}$ satisfies Riccati equation (65), (66) and additional condition (67).*

The Popov function

$$H(z) \triangleq \begin{bmatrix} B^{\mathrm{T}}(z^{-1}I - A^{\mathrm{T}})^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S^{\mathrm{T}} \\ S & R \end{bmatrix} \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix} =$$
$$R + S(zI - A)^{-1}B + B^{\mathrm{T}}(z^{-1}I - A^{\mathrm{T}})^{-1}S^{\mathrm{T}} + B^{\mathrm{T}}(z^{-1}I - A^{\mathrm{T}})^{-1}Q(zI - A)^{-1}B, \tag{68}$$

which can be associated with linear matrix inequality (59) and algebraic Riccati equation (65), (66), is the very convenient instrument for the examination of the algebraic Riccati equations and linear matrix inequalities solutions.

In [4] it was shown that in the case when the Popov function $H(z)$ has the full normal rank

$$\mathrm{normrank}\, H \triangleq \max_{\forall z \in \mathbb{C}} \mathrm{rank} H(z),$$

generalized inverse in nonlinear matrix inequality (63) and Riccati equation (65) turns to a normal inverse, and condition (66) fulfils automatically.

In particular, the Popov function $H(z)$ has the full normal rank in so-called *positive semidefinite case* when the inequality

$$\begin{bmatrix} Q & S^{\mathrm{T}} \\ S & R \end{bmatrix} \geqslant 0 \tag{69}$$

holds true.

The following theorem establishes the constructive sufficient condition for belonging of the solutions X of linear matrix inequality (59) to the set $\mathcal{L}_{\min}$.

Theorem 4 *Let in linear matrix inequality (59) the matrix pair (A,B) be stabilizable, the matrix $R > 0$, and positive semidefinite matrix $X = X^{\mathrm{T}} \in \Gamma$ be such that*

$$\operatorname{tr} X = \max_{\forall \widetilde{X} \in \Gamma} \operatorname{tr} \widetilde{X}.$$

Then the matrix X is the strongly rank-minimizing solution of linear matrix inequality (59), that is $X \in \mathcal{L}_{\min}$.

To proof Theorem 4, we need the following auxiliary lemma [12].

Lemma 4 *The matrix Lyapunov equation*

$$AYA^{\mathrm{T}} - Y + Q = 0 \tag{70}$$

with $Q = Q^{\mathrm{T}}$ has a unique symmetric solution $Y = Y^{\mathrm{T}}$ if and only if $\lambda_i \lambda_j \neq 1$ for all eigenvalues λ_i of the matrix A. At that

$$P = \sum_{k=0}^{\infty} A^k Q (A^{\mathrm{T}})^k > 0$$

if and only if the matrix A is asymptotically stable ($\rho(A) < 1$) and either $Q > 0$ (if $Q \geqslant 0$, then $P \geqslant 0$) or $Q = BB^{\mathrm{T}}$ and the matrix pair (A,B) is controllable.

Proof of Theorem 4 Consider the following optimization problem:

$$\begin{aligned} \operatorname{tr} X &\to \max, \\ L(X) &\geqslant 0. \end{aligned} \tag{71}$$

Since $R > 0$ and $X \geqslant 0$, $\operatorname{rank}(B^{\mathrm{T}}XB + R) = m$, the generalized inverse of the matrix $(B^{\mathrm{T}}XB + R)$ in inequality (63) and equation (65) is the normal inverse, and condition (66) and inequality (67) hold automatically. In this case, problem (71) is equivalent to the following optimization problem:

$$\operatorname{tr} X \to \max, \tag{72}$$

$$\mathrm{Ric}(X) \geqslant 0. \tag{73}$$

Let us form Lagrangian for problem (72), (73):

$$\mathcal{L}(X,Y) \triangleq \mathrm{tr}\{X + \mathrm{Ric}(X)Y\}, \tag{74}$$

where $Y \in \mathbb{R}^{n\times n}$ is the Lagrange multiplier matrix.

Necessary condition of optimality (stationarity of $\mathcal{L}(X,Y)$ with respect to X) requires that

$$\frac{\partial \mathcal{L}(X,Y)}{\partial X} = 0. \tag{75}$$

Find the derivative of Lagrangian (74) with respect to X :

$$\begin{aligned}\frac{\partial \mathcal{L}(X,Y)}{\partial X} &= \frac{\partial}{\partial X}\,\mathrm{tr}\,X + \frac{\partial}{\partial X}\,\mathrm{tr}\left\{A^{\mathrm{T}}XA - X - (A^{\mathrm{T}}XB + S^{\mathrm{T}})(B^{\mathrm{T}}XB + R)^{-1}\right.\\ &\quad \left.\times (B^{\mathrm{T}}XA + S) + Q\right\} = I + AYA^{\mathrm{T}} - Y - B(B^{\mathrm{T}}XB + R)^{-1}(B^{\mathrm{T}}XA + S)YA^{\mathrm{T}}\\ &\quad - B(B^{\mathrm{T}}XB + R)^{-1}(B^{\mathrm{T}}XA + S)Y(A^{\mathrm{T}}XB + S^{\mathrm{T}})(B^{\mathrm{T}}XB + R)^{-1}B^{\mathrm{T}}\\ &\quad + AY(A^{\mathrm{T}}XB + S^{\mathrm{T}})(B^{\mathrm{T}}XB + R)^{-1}B^{\mathrm{T}}\\ &= (A - (B^{\mathrm{T}}XB + R)^{-1}(B^{\mathrm{T}}XA + S))Y(A - (B^{\mathrm{T}}XB + R)^{-1}(B^{\mathrm{T}}XA + S))^{\mathrm{T}} - Y + I. \end{aligned} \tag{76}$$

Denote

$$K \triangleq -(B^{\mathrm{T}}XB + R)^{-1}(B^{\mathrm{T}}XA + S). \tag{77}$$

Putting expression (76) to zero, we obtain

$$(A + BK)Y(A + BK)^{\mathrm{T}} - Y + I = 0. \tag{78}$$

By virtue of Lemma 4 and theorem conditions, Lyapunov equation (78) has a unique symmetric solution $Y = \sum_{k=0}^{\infty}(A + BK)^k((A + BK)^{\mathrm{T}})^k > 0$.

Sufficient condition of optimality is given by

$$\mathrm{Ric}(X)Y = 0. \tag{79}$$

Since $\mathrm{Ric}(X) \geqslant 0$ and $Y > 0$, from equality (79) it follows that $\mathrm{Ric}(X) = 0$. Hence, by virtue of Lemma 3, the matrix X is the strongly rank-minimizing solution of linear matrix inequality (59). □

Remark 5 *Subject to condition (4) of Lemma 4, the problem of finding the strongly rank-minimizing solution of linear matrix inequality* $L(X) \geqslant 0$ *can be formulated as the following SDP-problem:*

$$\begin{gathered}\mathrm{tr}\,X \to \max,\\ L(X) \geqslant 0.\end{gathered} \tag{80}$$

3.2. Computing mean anisotropy of Gaussian signal via LMI

Consider the algebraic Riccati equation

$$ARA^{\mathrm{T}} - R - (ARC^{\mathrm{T}} + BD^{\mathrm{T}})(CRC^{\mathrm{T}} + DD^{\mathrm{T}})^{-1}(CRA^{\mathrm{T}} + DB^{\mathrm{T}}) + BB^{\mathrm{T}} = 0 \qquad (81)$$

arising from the random Gaussian sequence mean anisotropy computing problem. By virtue of identity (19),

$$CRC^{\mathrm{T}} + DD^{\mathrm{T}} \geqslant 0. \qquad (82)$$

Moreover, the matrix $CRC^{\mathrm{T}} + DD^{\mathrm{T}}$ in equation (81) is invertible.

Then the respective linear matrix inequality is given by

$$\widehat{L}(R) \triangleq \begin{bmatrix} ARA^{\mathrm{T}} - R + BB^{\mathrm{T}} & ARC^{\mathrm{T}} + BD^{\mathrm{T}} \\ CRA^{\mathrm{T}} + DB^{\mathrm{T}} & CRC^{\mathrm{T}} + DD^{\mathrm{T}} \end{bmatrix} \geqslant 0. \qquad (83)$$

Let $\widehat{\Gamma}$ be the solution set for linear matrix inequality (83). We denote the set of strongly rank-minimizing solutions of linear matrix inequality (83) as

$$\widehat{\mathfrak{L}}_{\min} = \left\{ R \in \widehat{\Gamma} : \operatorname{rank}\widehat{L}(R) = \operatorname{rank}(CRC^{\mathrm{T}} + DD^{\mathrm{T}}) \right\}.$$

The following theorem presents the formulas for computing the mean anisotropy of the sequence W generated by the filter with realization (15) using LMI-based approach.

Theorem 5 *Let a generating filter $G \in \mathcal{H}_2^{m\times m}$ have state-space realization (15) with A asymptotically stable and D nonsingular. Then the mean anisotropy (13) of the sequence $W = G * V$ is given by the formula*

$$\overline{\mathbf{A}}(G) = -\frac{1}{2}\log\det\left\{\frac{m(CRC^{\mathrm{T}} + DD^{\mathrm{T}})}{\operatorname{tr}(CPC^{\mathrm{T}} + DD^{\mathrm{T}})}\right\}, \qquad (84)$$

where the matrix $R \in \mathbb{R}^{n\times n}$ is the strongly rank-minimizing solution of linear matrix inequality (83) and the matrix P is the controllability gramian of the generating filter satisfying Lyapunov equation

$$APA^{\mathrm{T}} - P + BB^{\mathrm{T}} = 0. \qquad (85)$$

The proof follows immediately from Lemma 3 and Theorem 1.

Remark 6 *The problem of finding the solution $R \in \widehat{\mathfrak{L}}_{\min}$ of linear matrix inequality (83) and the solution P of Lyapunov equation (85) can be reduced to the following SDP-problem:*

$$\begin{gathered} \operatorname{tr}R \to \max, \quad \operatorname{tr}P \to \max, \\ \begin{bmatrix} \widehat{L}(R) & 0 \\ 0 & APA^{\mathrm{T}} - P + BB^{\mathrm{T}} \end{bmatrix} \geqslant 0. \end{gathered} \qquad (86)$$

To solve problem (86), one can use the existing SDP-software tools, for example, SEDUMI INTERFACE 1.04 [13].

3.3. Computing anisotropic norm via LMI

Consider the algebraic Riccati equation

$$A^{\mathrm{T}}RA - R - (A^{\mathrm{T}}RB + qC^{\mathrm{T}}D)(B^{\mathrm{T}}RB + qD^{\mathrm{T}}D - I_m)^{-1} \\ \times (B^{\mathrm{T}}RA + qD^{\mathrm{T}}C) + qC^{\mathrm{T}}C = 0, \quad q \in \left[0, \gamma^{-2}\right) \tag{87}$$

arising from the problem of computing the α-anisotropic norm $|\!|\!| F |\!|\!|_{\alpha}$ of system (37). According to Remark 3, for the admissible solution $R = R^{\mathrm{T}} \geqslant 0$ of Riccati equation (87), the inequality

$$B^{\mathrm{T}}RB + qD^{\mathrm{T}}D - I_m < 0 \tag{88}$$

holds true for $\forall q \in \left[0, \gamma^{-2}\right)$. Furthermore, the matrix $B^{\mathrm{T}}RB + qD^{\mathrm{T}}D - I_m$ in equation (87) is invertible. Then the respective linear matrix inequality is given by

$$\widetilde{L}(R) \triangleq \begin{bmatrix} A^{\mathrm{T}}RA - R + qC^{\mathrm{T}}C & A^{\mathrm{T}}RB + qC^{\mathrm{T}}D \\ B^{\mathrm{T}}RA + qD^{\mathrm{T}}C & B^{\mathrm{T}}RB + qD^{\mathrm{T}}D - I_m \end{bmatrix} \leqslant 0. \tag{89}$$

Let $\widetilde{\Gamma}$ be the solutions set for linear matrix inequality (89). We denote the set of strongly rank-minimizing solutions of linear matrix inequality (89) as

$$\widetilde{\mathfrak{L}}_{\min} = \left\{ R \in \widetilde{\Gamma} : \operatorname{rank} \widetilde{L}(R) = \operatorname{rank}(B^{\mathrm{T}}RB + qD^{\mathrm{T}}D - I_m) \right\}.$$

Theorem 6 *Let system (37) with asymptotically stable matrix A satisfying strict inequality (26) and the mean anisotropy level* α *of the input sequence W be given. Then there exists a unique pair* (q, R) *of the parameter* $q \in \left[0, \gamma^{-2}\right)$ *and the matrix* $R \in \mathbb{R}^{n \times n}$*, which is the strongly rank-minimizing solution of linear matrix inequality (89) such that*

$$-\frac{1}{2} \log\det \left\{ \frac{m\Sigma}{\operatorname{tr}(LPL^{\mathrm{T}} + \Sigma)} \right\} = \alpha \tag{90}$$

as

$$\Sigma = (I_m - qD^{\mathrm{T}}D - B^{\mathrm{T}}RB)^{-1}, \tag{91}$$

$$L = \Sigma(B^{\mathrm{T}}RA + qD^{\mathrm{T}}C), \tag{92}$$

where the matrix $P \in \mathbb{R}^{n \times n}$ *is the controllability gramian of generating filter*

$$G(z) = \left[\begin{array}{c|c} A + BL & B\Sigma^{1/2} \\ \hline L & \Sigma^{1/2} \end{array} \right], \tag{93}$$

satisfying Lyapunov equation

$$(A + BL)P(A + BL)^{\mathrm{T}} - P + B\Sigma B^{\mathrm{T}} = 0. \tag{94}$$

At that, filter (93) is the worst-case generating filter, and the α*-anisotropic norm of the system* F *is defined to be*

$$|||F|||_\alpha = \left\{\frac{1}{q}\left(1 - \frac{m}{\operatorname{tr}(LPL^{\mathrm{T}} + \Sigma)}\right)\right\}^{1/2}. \tag{95}$$

The proof follows immediately from Lemma 3 and Theorem 3.

Remark 7 *Thus, the algorithm for computing the* α*-anisotropic norm* $|||F|||_\alpha$ *of system (37) is similar to Algorithm 1 except point 1, at which, instead of finding the admissible solution* R *of algebraic Riccati equation (87), the following SDP-problem is to be solved:*

$$\begin{aligned} &\operatorname{tr} R \to \min, \\ &\widetilde{L}(R) \leqslant 0. \end{aligned} \tag{96}$$

4. Numerical examples

4.1. Mean anisotropy computation example

Consider the generating filter with transfer matrix $G(z) = (A, B, C, D)$, where

$$\begin{aligned}
A &= \begin{bmatrix} 0.2800 & 0.2442 & 0.0983 \\ -0.6568 & -0.1620 & 0.2307 \\ -0.0528 & -0.0920 & 0.9189 \end{bmatrix}, \\
B &= \begin{bmatrix} 1.9969 & 0.3630 \\ 0.6970 & -0.5670 \\ -1.3664 & -1.0442 \end{bmatrix}, \\
C &= \begin{bmatrix} -1.6282 & -0.4154 & 0.2294 \\ -1.1738 & 0.1751 & -1.2409 \end{bmatrix}, \\
D &= \begin{bmatrix} 0.7000 & 1.4548 \\ 0.4269 & -0.5102 \end{bmatrix}.
\end{aligned}$$

The matrix A spectral radius $\rho(A) = 0.8997 < 1$.

The mean anisotropy level of Gaussian sequence W generated by this filter calculated on the grounds of Theorem 1 equals to 0.97525562491097, and that for LMI-based approach of Theorem 5 using SDP-problem (86) equals to 0.97525561006214.

Let us note that for these two approaches, the absolute deviation of mean anisotropy levels is equal to $1.484883 \cdot 10^{-8}$ (the relative deviation equals to $1.522558 \cdot 10^{-6}\%$).)

4.2. Anisotropic norm computation example

Consider the linear discrete-time-invariant system with transfer matrix $F(z) = (A,B,C,D)$, where

$$A = \begin{bmatrix} 0.4501 & -0.0140 & -0.0435 \\ -0.2689 & 0.3913 & -0.4815 \\ 0.1068 & 0.2621 & 0.3214 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.0553 & 0.4218 & -0.0943 & -0.0897 \\ 0.1154 & 0.2382 & 0.4355 & 0.3936 \\ 0.2919 & -0.3237 & 0.4169 & -0.4421 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.3012 & 0.2468 & 0.4318 \\ -0.4847 & -0.0549 & -0.0340 \end{bmatrix},$$

$$D = \begin{bmatrix} -0.0814 & 0.0252 & 0.1721 & -0.4804 \\ 0.3462 & -0.2974 & 0.3381 & 0.1813 \end{bmatrix}.$$

The matrix A spectral radius $\rho(A) = 0.4962 < 1$.

Computation of $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of this system gives

$$\|F\|_2 = 0.9459, \quad \|F\|_\infty = 1.0591.$$

Results of calculating the α-anisotropic norm $|||F|||_\alpha$ for some values of the parameter α obtained via two approaches using Riccati equation (denoted as $|||F|||_\alpha^\dagger$) and linear matrix inequality (denoted as $|||F|||_\alpha^\ddagger$) with the relative accuracy $\varepsilon = 0.0001$ are presented in Table 1.

Figure 3 illustrates the dependence of the α-anisotropic norm $|||F|||_\alpha$ on the mean anisotropy level α. Figure 4 presents the graph of absolute deviation of the α-anisotropic norm values calculated via two different approaches.

5. Conclusion

In this paper, an approach to performance analysis of control systems is considered based on the notions of mean anisotropy of stationary Gaussian sequences and α-anisotropic norm of linear discrete-time-invariant systems.

Earlier presented method for computing the mean anisotropy in state-space is based on the solutions of algebraic Riccati and Lyapunov equations. This paper presents the method for computing the mean anisotropy, which reduces to the problem of finding the strongly rank-minimizing solution of linear matrix inequality, that is, to the semidefinite programming problem.

Table 1. Correspondence between mean anisotropy level of input signal and anisotropic norm of system

α	$\lvert\lvert\lvert F \rvert\rvert\rvert_\alpha^\dagger$	$\lvert\lvert\lvert F \rvert\rvert\rvert_\alpha^\ddagger$	α	$\lvert\lvert\lvert F \rvert\rvert\rvert_\alpha^\dagger$	$\lvert\lvert\lvert F \rvert\rvert\rvert_\alpha^\ddagger$	α	$\lvert\lvert\lvert F \rvert\rvert\rvert_\alpha^\dagger$	$\lvert\lvert\lvert F \rvert\rvert\rvert_\alpha^\ddagger$
0	0.4730	0.4730	0.1	0.5800	0.5799	2	0.9161	0.9161
0.01	0.5064	0.5064	0.2	0.6254	0.6254	3	0.9751	0.9752
0.02	0.5204	0.5204	0.3	0.6597	0.6597	4	1.0091	1.0090
0.03	0.5313	0.5313	0.4	0.6886	0.6887	5	1.0291	1.0291
0.04	0.5404	0.5404	0.5	0.7137	0.7137	6	1.0410	1.0410
0.05	0.5483	0.5483	0.6	0.7362	0.7362	7	1.0482	1.0481
0.06	0.5555	0.5556	0.7	0.7564	0.7565	8	1.0525	1.0525
0.07	0.5622	0.5622	0.8	0.7748	0.7748	9	1.0551	1.0550
0.08	0.5689	0.5690	0.9	0.7917	0.7916	10	1.0567	1.0568
0.09	0.5745	0.5745	1	0.8074	0.8074	11	1.0576	1.0577

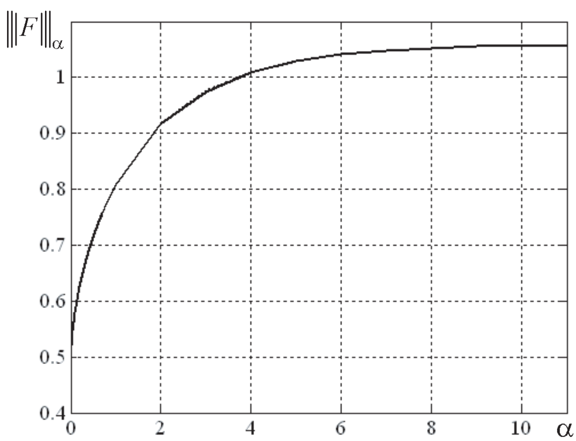

Figure 3. Dependence of the α-anisotropic norm $\lvert\lvert\lvert F \rvert\rvert\rvert_\alpha$ on the mean anisotropy level α

One of the earlier presented approaches for computing the anisotropic norm of system based on Newton's method for solving the nonlinear algebraic equation supposes solving the linked Riccati and Lyapunov equations at each step of iterative procedure. The method presented in this paper at each step of iteration supposes finding the strongly rank-minimizing solution of linear matrix inequality instead of solving the Riccati equation.

The developed methods for computing the mean anisotropy and anisotropic norm have shown the good accuracy and acceptable effectiveness in comparison with existing

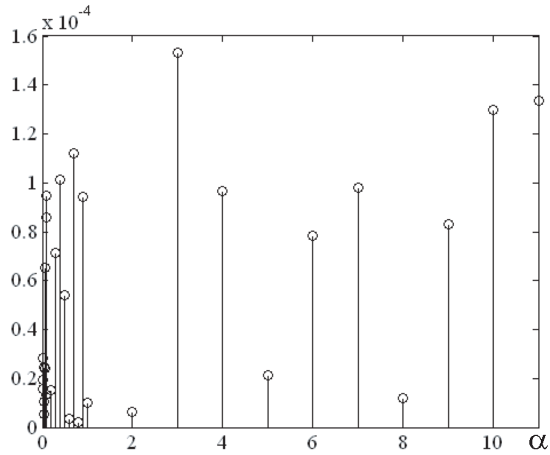

Figure 4. Absolute deviation of the α-anisotropic norm values calculated via two different approaches

approaches. The numerical examples of computing the mean anisotropy of Gaussian sequence and α-anisotropic norm of linear discrete-time-invariant system are presented in this paper.

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