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Output-reachability of positive linear discrete-time systems with delays

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The notion of output-reachability is extended for positive linear discrete-time systems with delays described by the state equations, by the transfer matrices and by the impulse matrices. Necessary and sufficient conditions for the output-reachability are established. The classical Cayley-Hamilton theorem is extended for coefficient matrices of the system impulse matrix. The considerations are illustrated by numerical examples.

Key words: linear, discrete-time, system, delay, output-reachability, positive system

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [2, 4]. Recent developments in positive systems theory and some new results are given in [5].

Recently, the reachability and minimum energy control of positive linear discrete-time systems with delays have been considered in [1, 7]. The output reachability of positive discrete-time systems without delays has been considered in [6].

In this paper the known [3, 6] notion of output-reachability will be extended for positive linear discrete-time systems with delays described by the state equations, by the transfer matrices and by the impulse matrices. For all these cases necessary and sufficient conditions for the output-reachability will be established. To the best knowledge of the

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author the output-reachability for the positive linear systems with delays has not been considered yet.

2. Systems described by state equations

Let $R^{n\times m}$ be the set of $n\times m$ real matrices with entries from the field of real numbers and $R^n = R^{n\times 1}$. The set $n\times m$ of matrices with real nonnegative entries will be denoted by $R_+^{n\times m}$ and $R_+^n = R_+^{n\times 1}$. The set of nonnegative integers will be denoted by Z_+ and the $n\times n$ identity matrix will be denoted by I_n. A matrix $A\in R_+^{n\times n}$ will be shortly called a positive one. Consider the discrete-time linear system

$$x(i+1) = A_0x(i) + A_1x(i-1) + Bu(i) \tag{1a}$$

$$y(i) = Cx(i) + Du(i) \tag{1b}$$

where $x(i)\in R^n$, $u(i)\in R^m$, $y(i)\in R^p$ are the state, input (control) and output vectors, respectively and $A_0, A_1 \in R^{n\times n}$, $B\in R^{n\times m}$, $C\in R^{p\times n}$, $D\in R^{p\times m}$.

The solution $x(i)$ of (1a) with the initial conditions

$$x_{-1} = x(-1), \quad x_0 = x(0) \tag{2}$$

has the form [1,7]

$$x(i) = \Phi(i)x_0 + \Phi(i-1)A_1x_{-1} + \sum_{k=0}^{i-1}\Phi(i-k-1)Bu(k) \tag{3}$$

where $\Phi(i)$ is the fundamental (transition) matrix satisfying the equation

$$\Phi(i+1) = A_0\Phi(i) + A_1\Phi(i-1) \tag{4}$$

with $\Phi(0) = I_n$, $\Phi(i) = 0$ for $i < 0$.

Definition 1. [1,4] *The system (1) is called (internally) positive if for every $x_0, x_{-1}\in R_+^n$ and all inputs sequences $u_i\in R_+^m$, $i\in Z_+$, $x(i)\in R_+^n$ and $y(i)\in R_+^p$, $i\in Z_+$.*

Theorem 1. [1,4] *The system (1) is positive if and only if*

$$A_0,\, A_1\in R_+^{n\times n}, \quad B\in R_+^{n\times m}, \quad C\in R_+^{p\times n}, \quad D\in R_+^{p\times m}. \tag{5}$$

Definition 2. *The positive system (1) is called output-reachable (shortly o-reachable) in q steps if for every $y_f\in R_+^p$ there exists a natural number q and an input sequence $\{u(0), u(1), \dots u(q-1)\}$ such that $y(q-1) = y_f$ for $x(0) = x(-1) = 0$.*

The matrix $A \in R^{n\times n}$ is called monomial if every its row and every its column has only one positive entry and its remaining entries are zero.

Theorem 2. *The positive system (1) is o-reachable in q steps if and only if one of the following conditions is satisfied:*

i) p linearly independent columns can be chosen from the matrix

$$R(q) := [C\Phi(q-2)B,\ CA\Phi(q-3)B, \ldots, CB, D] \quad (q \geqslant p) \tag{6}$$

so that the matrix R_p constructed from them is a monomial matrix;

ii) the inverse matrix R_p^{-1} has nonnegative entries, i.e.

$$R_p^{-1} \in R_+^{p\times p}. \tag{7}$$

PROOF. Substitution of (3) into (1b) yields

$$y(i) = C\Phi(i)x_0 + c\Phi(i-1)A_1x_{-1} + \sum_{k=0}^{i-1} C\Phi(i-k-1)Bu(k) + Du(i). \tag{8}$$

Using (8) for $i = q-1$ and $x_0 = x_{-1} = 0$ we obtain

$$\begin{aligned} y_f = y(q-1) &= \sum_{k=0}^{q-2} C\Phi(q-k-2)Bu(k) + Du(q-1) \\ &= [C\Phi(q-2)B, C\Phi(q-3)B, \ldots CB, D] \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(q-1) \end{bmatrix} = R(q)u_{0q} \end{aligned} \tag{9}$$

where

$$u_{0q} = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(q-1) \end{bmatrix}$$

From Definition 2 it follows that for every $y_f \in R_+^p$ there exists $u_{0q} \in R^{qm}$ if and only if the condition i) is satisfied. The equivalence of the conditions i) and ii) follows immediately from the fact that the inverse matrix of a positive matrix is positive if and only if it is a monomial matrix [4]. □

From Theorem 2 we have the following important corollary.

Corollary 1. *If the matrix $D \in R^{p\times m}$ $(m \geqslant p)$ contains p linearly independent monomial columns then the positive system (1) is o-reachable in one step for any matrices A_0, A_1, B, C.*

Theorem 3. *If*

$$\text{rank}C < p \quad and \quad D = 0 \tag{10}$$

then the positive system (1) is not o-reachable.

PROOF. If $D = 0$ then the matrix (6) can be written as the product of two matrices in the form

$$R(q) = C[\Phi(q-2)B, \Phi(q-3)B, \dots, B, 0]. \tag{11}$$

From (11) it follows that if $\text{rank}C < p$ then $\text{rank}R(q) < p$ and the condition i) is not satisfied. □

Example 1. Consider the positive system (1) with

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & a_3 \\ 1 & 0 & a_4 \\ 0 & 1 & a_5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{12}$$

$$(a_k \geqslant 0, \ \ k = 0, 1, \dots, 5)$$

for a) $D = [0 \ \ 0]^T$ and b) $D = [1 \ \ 0]^T$.

In the case a) the matrix (6) for $q = 4$ has the form

$$R(4) = [C\Phi(2)B, C\Phi(1)B, CB, D] = [C(A_0^2 + A_1)B, CA_0B, CB, D] = \begin{bmatrix} 1+a_3 & 0 & 0 & 0 \\ a_2+a_5 & 1 & 0 & 0 \end{bmatrix}$$

The conditions of Theorem 2 are not satisfied for any $a_k > 0$, $k = 2, 3, 4$ and the system is not o-reachable but it is o-reachable for $a_2 = a_3 = a_4 = 0$.

In the case b) we have

$$R(4) = [C\Phi(2)B, C\Phi(1)B, CB, D] = \begin{bmatrix} 1+a_3 & 0 & 0 & 1 \\ a_2+a_5 & 1 & 0 & 0 \end{bmatrix}. \tag{13}$$

The matrix (13) contains the monomial matrix $[C\Phi(1)B, D] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the system is o-reachable in three steps for any a_2, a_3 and a_5.

3. Systems described by transfer matrices

The transfer matrix of the system (1) has the form

$$T(z) = C[I_n z - A_0 - A_1 z^{-1}]^{-1} B + D. \tag{14}$$

From (4) we have

$$[I_n z - A_0 - A_1 z^{-1}]^{-1} = \sum_{k=0}^{\infty} \Phi(k) z^{-(k+1)}. \tag{15}$$

Substituting of (15) into (14) yields

$$T(z) = \sum_{k=0}^{\infty} C\Phi(k)Bz^{-(k+1)} + D = \sum_{k=0}^{\infty} T_k z^{-k} \tag{16}$$

where

$$T(z) = \begin{cases} D & \text{for} \quad k = 0 \\ C\Phi(k-1)B & \text{for} \quad k = 1, 2, \dots \end{cases} \tag{17}$$

Theorem 4. *The positive system with transfer matrix $T(z)$ is o-reachable in q steps if and only if the matrix*

$$\bar{R}(q) = [T_0, T_1, \dots, T_{q-1}] \tag{18}$$

contains p linearly independent monomial columns.

PROOF. Substituting (17) into (18) we obtain

$$\bar{R}(q) = [D,\ CB,\ C\Phi(1)B, \dots, C\Phi(q-2)B]. \tag{19}$$

The matrices (6) and (19) contain the same linearly independent monomial columns. Therefore, by Theorem 2 the positive system with transfer matrix $T(z)$ is o-reachable in q steps if and only if the matrix (18) contains p linearly independent monomial columns. □

To compute the matrices T_k, $k = 0, 1, \dots$ for a given transfer matrix $T(z)$ the following Lemma can be used.

Lemma *Let*

$$T(z) = \frac{N_n z_n + N_{n-1} z^{n-1} + \dots + N_1 z + N_0}{z^n - a_{n-1} z^{n-1} - \dots - a_1 z - a_0}, \quad N_i \in R^{p \times m}, \quad i = 0, 1, \dots, n. \tag{20}$$

Then

$$T_0 = N_n, \quad T_1 = N_{n-1} + a_{n-1} T_0, \dots, T_n = N_0 + a_{n-1} T_{n-1} + \dots + a_1 T_1 + a_0 T_0 \tag{21}$$

and

$$T_{n+k} = a_0 T_k + a_1 T_{k+1} + \dots + a_{n-1} T_{n+k-1} \ \text{ for } \ k = 1, 2, \dots \tag{22}$$

PROOF. From (20) and (16) we have

$$N_n z_n + N_{n-1} z^{n-1} + \dots + N_1 z + N_0 = (z^n - a_{n-1} z^{n-1} - \dots - a_1 z - a_0)(T_0 + T_1 z^{-1} + T_2 z^{-2} + \dots) \tag{23}$$

Comparing the coefficients at the same power of z of the equality (23) we obtain (21) and (22). □

Note that from (22) for $k = 1$ we have

$$T_{n+1} = a_0 T_1 + a_1 T_2 + \dots a_{n-1} T_n \tag{24}$$

The equality (24) is an extension of the classical Cayley-Hamilton theorem for the matrices T_k.

Example 2. Consider the positive system with the transfer matrix

$$T(z) = \frac{1}{z^6 - z^5 - 2z^3 + 2z^2 - 2} \begin{bmatrix} 2z^3 - 2z^2 \\ z^6 - z^5 - 2z^3 + 2z^2 + 2z - 2 \end{bmatrix}. \tag{25}$$

In this case

$$d(z) = z^6 - z^5 - 2z^3 + 2z^2 - 2,$$
$$N(z) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} z^6 + \begin{bmatrix} 0 \\ -1 \end{bmatrix} z^5 + \begin{bmatrix} 2 \\ -2 \end{bmatrix} z^3 + \begin{bmatrix} -2 \\ 2 \end{bmatrix} z^2 + \begin{bmatrix} 0 \\ 2 \end{bmatrix} z + \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

Using (21) and (18) for $q = 4$ we obtain

$$T_0 = N_6 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; T_1 = N_5 + a_5 T_0 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; T_2 + N_4 + a_5 T_1 + a_4 T_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_3 = N_3 + a_5 T_2 + a_4 T_1 + a_3 T_0 = \begin{bmatrix} 2 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

and

$$\bar{R}(4) = [T_0 \; T_1 \; T_2 \; T_3] = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{26}$$

The matrix (26) contains the monomial matrix $[T_0 \;\; T_3] = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ and by Theorem 4 the system with transfer matrices (25) is o-reachable in four steps.

A positive realization of (25) has the form [4, 7]

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{27}$$

For (27) the matrix (6) for $q = 4$ is equal to

$$R(4) = [C\Phi(2)B, C\Phi(1)B, CB, D] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

and by Theorem 2 the positive system with the matrices (27) is o-reachable in four steps.

4. Systems described by impulse matrices

Consider a positive linear system described by an impulse matrix $g(i) \in R_+^{p\times m}$, $i \in Z_+$. It is well-known [3,4] that the output $y(i)$ for a given input sequence $u(i)$, $i \in Z_+$ and zero initial conditions is given by the formula

$$y(i) = \sum_{k=0}^{i} g(i-k)u(k) = \sum_{k=0}^{i} g(k)u(i-k). \tag{28}$$

The impulse matrix is related with the transfer matrix of the system by the equality [3, 4]

$$g(i) = Z^{-1}[T(z)] \tag{29}$$

where Z^{-1} is the inverse operator of the $\mathcal{Z}$-transform. From (29) and (17) it follows that

$$g(i) = T_i = \begin{cases} D & \text{for} \quad i = 0 \\ CA^{(i-1)}B & \text{for} \quad i = 1, 2, \dots \end{cases} \tag{30}$$

Theorem 5. *The positive system described by an impulse matrix $g(i) \in R_+^{p\times m}$ is o-reachable in q steps if and only if the matrix*

$$\hat{R}(q) = [g(0)\ g(1)\ \dots\ g(q-1)] \tag{31}$$

contains p linearly independent monomial columns.

The proof follows immediately from relations (30) and Theorem 2.

Let the positive system with a given impulse matrix $g(i)$ be o-reachable in q steps. An input sequence $u(0), u(1), \dots, u(q-1)$ that steers the output system from zero initial state to the desired final value $y(q-1) = y_f$ can be computed as follows. From (28) and (31) we have

$$y(q-1) = y_f = [g(q-1)\ g(q-2)\ \dots\ g(0)] \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(q-1) \end{bmatrix}. \tag{32}$$

From the matrix $[g(q-1)\ g(q-2)\ \dots\ g(0)]$ we choose p linearly independent monomial columns $g_1, g_2, \dots, g_p$ and we built from them the monomial matrix

$$R_p = [g_1, g_2, \dots, g_p]. \tag{33}$$

Let $u_1, u_2, \ldots, u_p$ be the components of the matrix $[u(0)^T, u(1)^T, \ldots, u(q-1)^T]^T$ that correspond to the columns of the matrix (33) and its remaining components be zero. Then from (32) we have

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix} = R_p^{-1} y_f. \tag{34}$$

Using (34) we may compute the desired sequence $u(0)^T, u(1)^T, \ldots, u(q-1)$ that steers the output system to the final value y_f. Note that if $m > 1$ then usually there exist many input sequences that steer the output system from zero initial state to the desired y_f.

Example 3. Check the o-reachability of the positive system with the impulse matrix

$$g(i) = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{for} \quad i = 0 \\ \begin{bmatrix} 0 \\ (0.2)^i \end{bmatrix} & \text{for} \quad i = 1, 2, \ldots \end{cases} \tag{35}$$

and find the input sequence that steers the output system from zero state to $y_f = [1\ 1]^T$. Using (31) for $q = 2$ we obtain the monomial matrix

$$\hat{R}(2) = [g(0)\ g(1)] = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}. \tag{36}$$

Therefore, the positive system is o-reachable in two steps. Using (32) and (36) we obtain

$$y(1) = y_f = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [g(0)\ g(1)] \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

and

$$\begin{bmatrix} u_2(1) \\ u_1(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

In this case the input sequence is unique.

5. Concluding remarks

The well-known notion of output-reachability has been extended for positive linear discrete-time systems with delays described by state equations, by the transfer matrices and by the impulse matrices.

Necessary and sufficient conditions for the output-reachability for all cases of description of the systems have been established. A method for checking of the output-reachability has been proposed. The classical Cayley-Hamilton theorem has been extended for the T_k, $k = 0, 1, \ldots$ matrices. A method for computation of the input sequences steering the output system from zero initial state to the desired value y_f has been given. The considerations have been illustrated by numerical examples. The considerations can be easily extended for linear discrete-time positive systems with many delays. An extension of these considerations for positive linear 1-D and 2-D [3, 4] systems with delays is also possible.

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