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On the admissible perturbations for discrete systems

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We consider a discrete system described by $x_{i+1} = Ax_i,\ i \geqslant 0$ with the output function $y_i = Cx_i,\ i \geqslant 0$ which is subject to the constraints $y_i \in \Omega, \forall i \geqslant 0$, where Ω is given subset of $\mathbb{R}^q$. Then we investigate the admissible nonlinear perturbations $(N_i)_i$, i.e., the ones such that the corresponding perturbed output function $y_i^p = Cx_i^p$ where $x_{i+1}^p = (A + N_i)x_i^p,\ i \geqslant 0$ remains in the constraints set Ω for all $i \geqslant 0$.

Key words: admissible perturbations, discrete systems, discrete delayed systems, nonlinear perturbations

1. Introduction

The natural connections between a system and its environnement necessarily involves, in the mathematical modelling, the presence of certain undesirable parameters. Facing such problems, scientists developed different approaches in systems theory such as the identifiability ([11], [7]), the sentinels theory ([8], [9]), the detectability [1], the H^∞-control theory ([5], [6]), the filtering theory ([2], [3], [12]) and the admissible sets theory ([10], [4]).

In this paper we develop a technique which allows to determine among a class of disturbances, those which are relatively tolerable. More precisely, let's Consider the discrete-time system

$$(S) \begin{cases} x_{i+1} = Ax_i\ , \quad i \geqslant 0 \\ x_0 \quad \text{is given} \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^n$, and the operator A is a $n \times n$ matrix. The corresponding output is

$$y_i = Cx_i\ , \ \ i \geqslant 0$$

where $y_i \in \mathbb{R}^q$ and C is a $q \times n$ matrix. Motivated by practical considerations, we suppose that the output is subject to constraints summarised by

$$y_i \in \Omega\ , \ \ \forall i \geqslant 0$$

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where Ω is an appropriate subset of $\mathbb{R}^q$. Subject to perturbations, the real version of the output function is

$$y_i^p = Cx_i^p \ , \quad i \geqslant 0 \tag{2}$$

where $(x_i^p)_i$ is the perturbed state given by

$$(S_p) \left\{ \begin{array}{rcl} x_{i+1}^p & = & (A+N_i)x_i^p \ , \quad i \geqslant 0 \\ x_0^p & = & x_0 + \omega \end{array} \right. \tag{3}$$

$(N_i)_{i\geqslant 0}$ are a nonlinear maps which describe the disturbances on the dynamic A and ω is a perturbation which enters the initial state x_0.

It is reasonable to agree that a disturbance $(\omega, (N_i)_i)$ is tolerable if the resulting output function $(y_i^p)_{i\geqslant 0}$ verify

$$y_i^p \in \Omega \ , \quad \forall i \geqslant 0.$$

In all this work, we suppose that the disturbances $(N_i)_i$ susceptible of infecting the system (S) are only persistent on a given time interval $\{0,\ldots,I\}$. i.e.,

$$N_i \equiv 0 \ , \quad \forall i \geqslant I+1.$$

I is called the age of the disturbances $(N_i)_i$.

2. Characterization of the admissible perturbations

An admissible perturbation is a sequence of maps, Θ say, which belongs to the set E given by

$$E = \{\Theta = (\omega, (N_i)_{0\leqslant i\leqslant I}) \ / \ y_i^p \in \Omega \ , \ \forall i \geqslant 0\}$$

where y_i^p is the output function described by (2). In order to characterize an admissible perturbation Θ we rewrite the set E as follows

$$E = V \cap W \tag{4}$$

where

$$V = \{\Theta \ / \ y_i^p \in \Omega \ , \ \forall i \in \{0,\ldots,I+1\}\}$$
$$W = \{\Theta \ / \ y_i^p \in \Omega \ , \ \forall i \geqslant I+2\}.$$

While the set V is characterized by a finite number of equations the set W is described by infinite ones. To overcome this difficulty we will establish suffucient conditions which allow us to characterize the set W by a finite number of equations.

It follows from (3) that

$$\begin{array}{rcl} x_i^p & = & (A+N_{i-1})\ldots(A+N_0)x_0^p \\ & = & \displaystyle\prod_{k=1}^{i}(A+N_{i-k}).x_0^p \end{array} .$$

Since the perturbation $(N_i)_i$ has a finite age I, we obtain for all $i \geqslant I+2$,

$$\begin{aligned} x_i^p &= A^{i-I-1} \prod_{k=i-I}^{i} (A+N_{i-k}).x_0^p \\ &= A^{i-I-1} \prod_{j=0}^{I} (A+N_{I-j}).(x_0+\omega) \end{aligned}$$

hence

$$x_i^p = A^{i-I-2}\Lambda(\Theta) \tag{5}$$

where $\Lambda(\Theta)$ is the vector given by

$$\Lambda(\Theta) = A\prod_{j=0}^{I}(A+N_{I-j}).(x_0+\omega). \tag{6}$$

Consequently, the set W can be written as follows

$$\begin{aligned} W &= \{\Theta \;/\; CA^{i-I-2}\Lambda(\Theta) \in \Omega \;,\; \forall i \geqslant I+2\} \\ &= \{\Theta \;/\; CA^{j}\Lambda(\Theta) \in \Omega \;,\; \forall j \geqslant 0\}. \end{aligned}$$

This suggest us to consider the set Z given by

$$Z = \{x \in \mathbb{R}^n \;/\; CA^j x \in \Omega \;,\; \forall j \geqslant 0\}. \tag{7}$$

For each integer k we define the set Z_k by

$$Z_k = \{x \in \mathbb{R}^n \;/\; CA^j x \in \Omega \;,\; \forall j \in \{0,\ldots,k\}\}$$

then we have the following result

Proposition 1 *Suppose that $Z_k = Z_{k+1}$ for some integer k, then the set Z given by* (7) *is described by a finite number of equations, more precisely we have $Z = Z_k$. Conversely, if $Z = Z_k$ for some integer k then $Z_k = Z_{k+1} = Z_j,\ \forall j \geqslant k$.*

Proof.
Suppose that there exists an integer k such that $Z_k = Z_{k+1}$, then we have

$$Ax \in Z_k \quad , \quad \forall x \in Z_k$$

and by iteration

$$A^j x \in Z_k \;,\; \forall j \geqslant 1, \forall x \in Z_k$$

hence $Z_k \subset Z$. But Z is a subset of Z_k, thus $Z = Z_k$. Conversely, if $Z_k = Z$ for some integer k then we deduce that $Z_k \subset Z_{k+1}$ which implies that $Z_k = Z_{k+1}$. ■

Since the set W is given by

$$W = \{\Theta \;/\; \Lambda(\Theta) \in Z\}$$

it follows that the characterization of the set Z by a finite number of equations implies a well characterization of the set W.

Proposition 2 *If $Z = Z_k$ for some integer k then the set of all admissible perturbations is characterized by*

$$E = \{\Theta \;/\; y_i^p \in \Omega \;,\; \forall i \in \{0, \dots, I+2+k\}\}.$$

Proof.
From the hypothesis it follows that

$$\begin{aligned} W &= \{\Theta \;/\; \Lambda(\Theta) \in Z_k\} \\ &= \{\Theta \;/\; CA^j \Lambda(\Theta) \in \Omega \;,\; \forall j \in \{0, \dots, k\}\} \\ &= \{\Theta \;/\; CA^{i-I-2} \Lambda(\Theta) \in \Omega \;,\; \forall i \in \{I+2, \dots, I+2+k\}\} \end{aligned}$$

then we use (5) to establish that

$$W = \{\Theta \;/\; y_i^p \in \Omega \;,\; \forall i \in \{I+2, \dots, I+2+k\}\}$$

and by equation (4) we deduce that $E = \{\Theta \;/\; y_i^p \in \Omega \;,\; \forall i \in \{0, \dots, I+2+k\}\}$. ■

In order to determine the smallest integer k_*, if there exists, such that $Z = Z_{k_*}$ we suggest the following algorithm.

Algorithm 1 :

step 1 : Set $k = 0$
step 2 : If $Z_k = Z_{k+1}$ then set $k_* = k$ and stop,
else continue.
step 3 : Replace k by $k+1$ and return to step 2.

To show how the test $Z_k = Z_{k+1}$ of step 2 can be implemented we consider the case where the set Ω is defined by

$$\Omega = \{y \in \mathbb{R}^q \;/\; f_j(y) \leqslant 0 \;,\; j = 1, \dots, s\}$$

where $f_j : \mathbb{R}^q \to \mathbb{R}$ are a given functions, such a sets have many importance in a practical view. In this case, the set Z_k is described by

$$Z_k = \{x \in \mathbb{R}^n \;/\; f_j(CA^i x) \leqslant 0 \;,\; j = 1, \dots, s \;,\; i = 0, \dots, k\}.$$

Since, for every integer k we have $Z_{k+1} \subset Z_k$, it follows that the test $Z_k = Z_{k+1}$ is true if and only if $Z_k \subset Z_{k+1}$, or equivalently

$$f_j(CA^{k+1}x) \leqslant 0 \;,\; \forall j \in \{1, \dots, s\} \;,\; \forall x \in Z_k$$

or

$$\sup_{x \in Z_k} f_j(CA^{k+1}x) \leqslant 0 \;,\; \forall j \in \{1, \dots, s\}.$$

or

$$\sup_{\begin{cases} f_l(CA^i x) \leqslant 0 \\ l \in \{1,\dots,s\} \\ i \in \{0,\dots,k\} \end{cases}} f_j(CA^{k+1}x) \leqslant 0 \ , \ \forall j \in \{1,\dots,s\}.$$

Hence algorithm 1 can be implemented as follows

Algorithm 2

step 1 : Let $k := 0$;

step 2 : For $j = 1,\dots,s$, do :

Maximize $J_j(x) = f_j(CA^{k+1}x)$

$$\begin{cases} f_l(CA^i x) \leqslant 0, \\ l \in \{1,\dots,s\}, \\ i \in \{0,\dots,k\}. \end{cases}$$

Let J_j^* be the calculated maximum value of J_j.

If $J_j^* \leqslant 0$, for $j = 1,\dots,s$ then

set $k_* := k$ and stop.

Else continue.

step 3 : Replace k by $k+1$ and return to step 2.

The success of algorithm 2 depends on the existence of effective algorithms for solving the rather large mathematical programming problems which arise. This presents some difficulty because global optima are needed. But when Ω is a polyhedron (i.,e., the functions f_j are affine for all j), the programming problems are linear and the difficulty disappears.

It is clear that algorithm 2 will converge if and only if there exists an integer k such that $Z = Z_k$. Sufficient conditions for the convergence of algorithm 2 are given by the following proposition.

Proposition 3 *If the following hypothesis are verified*
1) The pair (C,A) is observable (i.e., the matrix $[C^T, A^TC^T,\dots,(A^T)^{n-1}C^T]$ has rank n)
2) A is asymptotically stable (i.e., $|\lambda| < 1$ for every λ in the spectrum of A)
3) Ω is bounded
4) The origine belongs to the interior of Ω.
Then there exists an integer k such that $Z = Z_k$.

Proof.

Define the operator H by

$$H = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

then it follows from the definition of Z_{n-1} that

$$Hx \in \overbrace{\Omega \times \ldots \times \Omega}^{n\text{- times}} \;,\;\; \forall x \in Z_{n-1}. \tag{8}$$

On the other hand, the observability of the pair (C, A) implies that the matrix $H^T H$ is invertible, consequently there exists a constant $\alpha > 0$ such that

$$\alpha\|x\|^2 \leqslant < H^T Hx, x > \;\; , \forall x \in \mathbb{R}^n$$

which implies that

$$\alpha\|x\|^2 \leqslant \|H^T\|\|Hx\|\|x\|.$$

Since Ω is bounded, from equation (8) we deduce that

$$\alpha\|x\|^2 \leqslant \beta\|x\| \;\; , \forall x \in Z_{n-1}$$

where β is an appropriate constant. So $\|x\| \leqslant \gamma$, $\forall x \in Z_{n-1}$ for some constant $\gamma > 0$. Hence

$$Z_{n-1} \subset B(0, \gamma) = \{x \in \mathbb{R}^n \; / \; \|x\| \leqslant \gamma\}.$$

Now, since $0 \in interior(\Omega)$ there exists a real $\epsilon > 0$ such that $B(0, \epsilon) \subset \Omega$. The asymptotic stability of A implies the existence of an integer $k_0 \geqslant n - 1$ such that

$$\|CA^{k_0+1}\| \leqslant \frac{\epsilon}{\gamma}.$$

For every $x \in Z_{k_0}$ we have

$$\|CA^{k_0+1}x\| \leqslant \|CA^{k_0+1}\|\|x\|$$

but $Z_{k_0} \subset Z_{n-1} \subset B(0, \gamma)$, so we have

$$\|CA^{k_0+1}x\| \leqslant \epsilon \;\; , \forall x \in Z_{k_0}.$$

Consequently, $CA^{k_0+1}x \in B(0, \epsilon) \subset \Omega$, $\forall x \in Z_{k_0}$. Thus $Z_{k_0} \subset Z_{k_0+1}$, or equivalently $Z_{k_0} \subset Z_{k_0+1}$. ■

3. Example

Consider system (S) with

$$A = \begin{pmatrix} 1.6 & -0.7 \\ 4.2 & -1.9 \end{pmatrix} \quad \text{and} \quad C = (3\ 4)$$

The constraints set Ω is given by $\Omega = [-0.3, 0.3]$. Here we suppose that the age of the perturbations is $I = 10$. We verify that the matrix A is asymptotically stable and the pair (C, A) is observable, thus by theorem 3 it follows that algorithm 2 is convergent. Indeed, application of this algorithm shows that $k_* = 3$. Consequently, the only perturbations Θ which didn't affect seriously system (S) are those which verify the following equations

$$|y_i^p| \leqslant 0.3 \ , \ \forall i \in \{0, \dots, 15\}.$$

An example of admissible perturbation is the following

$$\omega = \begin{pmatrix} 0.01 \\ 0.01 \end{pmatrix} \quad , N_i : \begin{pmatrix} x \\ y \end{pmatrix} \hookrightarrow \begin{pmatrix} x.y/(i+1) \\ x^2/(i+1) \end{pmatrix} \quad \forall i \in \{0, \dots, 10\}$$

In the following section we show that the results obtained in this section can be extended to discrete-time delayed systems.

4. Discrete-time delayed systems

Consider the discrete delayed systems described by

$$(S_d) \begin{cases} x_{i+1} & = \displaystyle\sum_{j=0}^{r} A_j x_{i-j} \ , \ i \geqslant 0 \\ x_0 & \text{given} \\ x_k & = \alpha_k \ , \ -r \leqslant k \leqslant -1 \end{cases} \tag{9}$$

and the delayed output function

$$y_i = \sum_{j=0}^{d} C_j x_{i-j} \ , \ i \geqslant 0 \tag{10}$$

where $x_i \in I\!R^n$ is the state variable, $(A_j)_j$ are matrces of order n and $(C_j)_j$ a $q \times n$ matrices. Moreover we suppose that the trajectory of (S_d) satisfies the constraints summarized by

$$y_i \in \Omega \ , \ \forall i \geqslant 0. \tag{11}$$

The presence of perturbations on the dynamics and initial conditions of (S_d) affect the output function y_i. That is the real output function is given by

$$y_i^p = \sum_{j=0}^{d} C_j x_{i-j}^p \ , \ i \geqslant 0$$

where $(x_i^p)_i$ is the solution of the perturbed system described by

$$(S_d^p) \quad \begin{cases} x_{i+1}^p &= \sum_{j=0}^{r}(A_j + N_{i,j})x_{i-j}^p \ , \ i \geqslant 0 \\ x_0^p &= x_0 + \omega_0 \\ x_k^p &= \alpha_k + \omega_k \ , \ -r \leqslant k \leqslant -1 \end{cases} \tag{12}$$

where $(N_{i,j})_{i,j}$ are nonlinear functions and $\omega_{-r}, \ldots, \omega_0 \ \in \mathbb{R}^n$ are perturbations which affect the initial conditions. Here we suppose that the perturbations have a finite age I, i.e.,

$$N_{i,j} \equiv 0 \ , \ \forall i \geqslant I+1 \quad , \quad \forall j \in \{0, \ldots, r\}.$$

Then we investigate the admissible ones, i.e., the perturbations such that the perturbed output funcion satisfies also the consraints, i.e.,

$$y_i^p \in \Omega \ \ , \ \forall i \geqslant 0.$$

4.1. A state space technique

As above define the set E_d of all admissible perturbations by

$$E_d = \hat{V} \cap \hat{W} \tag{13}$$

where

$$\begin{aligned} \hat{V} &= \{\Theta_d = (\omega_{-r}, \ldots, \omega_0, (N_{i,j})_{0 \leqslant i \leqslant I; j}) \ / \ y_i^p \in \Omega \ \ , \ \forall i \in \{0, \ldots, I+1\}\} \\ \hat{W} &= \{\Theta_d = (\omega_{-r}, \ldots, \omega_0, (N_{i,j})_{0 \leqslant i \leqslant I; j}) \ / \ y_i^p \in \Omega \ \ , \ \forall i \geqslant I+2\} \end{aligned}$$

and define the variables $(e_i)_{i \geqslant 0}$ and $(e_i^p)_{i \geqslant 0}$ by

$$\begin{aligned} e_i &= (x_i, x_{i-1}, \ldots, x_{i-r})^T \\ e_i^p &= (x_i^p, x_{i-1}^p, \ldots, x_{i-r}^p)^T \end{aligned}$$

where $(x_i)_{i \geqslant 0}$ and $(x_i^p)_{i \geqslant 0}$ are respectively the solutions of (S_d) and (S_d^p). Then we have the following result.

Proposition 4 *$(e_i)_{i \geqslant 0}$ and $(e_i^p)_{i \geqslant 0}$ are respectively the solutions of the following systems*

$$(\hat{S}) \ \begin{cases} e_{i+1} &= \Phi e_i \ , \ i \geqslant 0 \\ e_0 &= (x_0, \alpha_{-1}, \ldots, \alpha_{-r})^T \end{cases} \tag{14}$$

$$(\hat{S}_p) \ \begin{cases} e_{i+1}^p &= (\Phi + \hat{N}_i)e_i^p \ , \ i \geqslant 0 \\ e_0^p &= e_0 + \omega \end{cases} \tag{15}$$

where Φ *and* $\hat{N}_i$ *are given by*

$$\Phi = \begin{bmatrix} A_0 & A_1 & \dots & A_r \\ I & 0 & \dots & 0 \\ & & \ddots & \\ 0 & & I & 0 \end{bmatrix} \in \mathcal{L}(\mathbb{R}^{(r+1)n}) \ , \ \hat{N}_i = \begin{bmatrix} N_{i,0} & N_{i,1} & \dots & N_{i,r} \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

and $\omega = (\omega_0, \alpha_{-1}, \dots, \alpha_{-r})^T$.

Proof.
From system (S_d) it follows that

$$x_{i+1} = (A_0, A_1, \dots, A_r)e_i$$

on the other hand we have

$$\begin{aligned} x_i &= (I, 0, \dots, 0)e_i \\ &\vdots \\ x_{i+1-r} &= (0, \dots, I, 0)e_i \end{aligned}$$

hence $e_{i+1} = \Phi e_i \ , \ \forall i \geqslant 0$. ■

The output function y_i and y_i^p can be written in terms of the new state variables e_i and e_i^p as follows

$$y_i = \hat{C}e_i \ , \ y_i^p = \hat{C}e_i^p \ , \ \forall i \geqslant 0 \tag{16}$$

where $\hat{C}$ is the $q \times (r+1)n$ matrix given by

$$\hat{C} = (C_0, \dots, C_d, 0, \dots, 0).$$

Now since (S_d) and $(\hat{S})$ have the same output function y_i and the same perturbed output function y_i^p, we deduce that the perturbation $\Theta_d = (\omega_{-r}, \dots, \omega_0, (N_{i,j})_{i,j})$ of system (S_d) is admissible if and only if the perturbation $\hat{\Theta}_d = (\omega, (\hat{N}_i)_i)$ of system $(\hat{S})$ is admissible. But $(\hat{S})$ is a discrete system without delay, so we can apply results of section 2 to characterize the admissible perturbations of $(\hat{S})$, that means

Proposition 5 *If the following hypothesis are satisfied*
1) the pair $(\hat{C}, \Phi)$ *is observable.*
2) Φ *is asymptotically stable.*
3) The set Ω *is bounded.*
4) The origine is in the interior of Ω.
Then there exists an integer k *such that* $E_d = \{\Theta_d \ / \ y_i^p \in \Omega \ , \ \forall i \in \{0, \dots, I+2+k\}\}$

Remark 6 *The determination of the integer* k *cited in proposition 5 is obtained by applying algorithm 2 with the changes* $C \to \hat{C}$ *and* $A \to \Phi$.

In the following, we will establish sufficient conditions under which the set E_d is given by

$$E_d = \{\Theta_d \;/\; y_i^p \in \Omega \;\;,\; \forall i \in \{0, \ldots, I+2+r\}\} \tag{17}$$

where r is the number of delays in the state variable. For this we consider two cases.
a) In the first case we suppose that $\mathrm{Dim}\,\Omega = n$ (i.e., the observation space and the state space have the same dimension).
b) In the second case we suppose that $\mathrm{Dim}\,\Omega = q < n$.

First case, Dim $\Omega = n$.
In this case every C_i is an $n \times n$ matrix.

Proposition 7 *Suppose the following assumptions to hold*
i) C_i *commutes with* A_j *for every* $i \in \{0, \ldots, d\}$ *and every* $j \in \{0, \ldots, r\}$
ii) $\displaystyle\sum_{i=0}^{r} A_i\Omega \subset \Omega$
where $\displaystyle\sum_{i=0}^{r} A_i\Omega = \{A_0 z_0 + A_1 z_1 + \ldots + A_r z_r \mid z_i \in \Omega, 0 \leqslant i \leqslant r\}$. *Then the set of all admissible perturbations is characterized by* (17).

Proof.
Applying the techniques of section 2 we have

$$E_d = \hat{V} \cap \hat{W}$$

where

$$\begin{array}{rcl} \hat{V} & = & \{\Theta_d \;/\; y_i^p \in \Omega \;\;,\; \forall i \in \{0, \ldots, I+1\}\} \\ \hat{W} & = & \{\Theta_d \;/\; y_i^p \in \Omega \;\;,\; \forall i \geqslant I+2\}\} \end{array} .$$

We have also

$$\hat{W} = \{\Theta_d \;/\; \bar{\Lambda}(\hat{\Theta}_d) \in \hat{Z}\}$$

where

$$\hat{Z} = \{e \in I\!\!R^{(r+1)n} \;/\; \hat{C}\Phi^i e \in \Omega, \; \forall i \geqslant 0\}$$

and

$$\bar{\Lambda}(\hat{\Theta}_d) = \Phi \prod_{j=0}^{I} (\Phi + \hat{N}_{I-j})(e_0 + \omega).$$

To prove the proposition it suffices to show that $\hat{Z}_r = \hat{Z}_{r+1}$, where

$$\hat{Z}_k = \{e \in I\!\!R^{(r+1)n} \;/\; \hat{C}\Phi^i e \in \Omega, \; \forall i \in \{0, \ldots, k\}\}.$$

We have

$$\hat{C}\Phi^i z = y_i(z) \;,\; \forall i \in \{0, \ldots, r\} \;\; \forall z = (z_0, \ldots, z_r) \in I\!\!R^{(r+1)n}$$

where $y_i(z)$ is the output function of system (14) with initial state $e_0 = z$. Consequently, if $z \in \hat{Z}_r$ then

$$y_i(z) \in \Omega \ , \ \forall i \in \{0, \ldots, r\}. \tag{18}$$

On the other hand,

$$y_{r+1}(z) = \sum_{j=0}^{d} C_j x_{r+1-j}(z)$$

where $(x_i(z))_i$ is the solution of system (9) with initial conditions $x_0 = z_0, x_{-1} = z_1, \ldots, x_{-r} = z_r$. Hence,

$$\begin{aligned} y_{r+1}(z) &= \sum_{j=0}^{d} C_j \sum_{k=0}^{r} A_k x_{r-j-k}(z) \\ &= \sum_{k=0}^{r} A_k \sum_{j=0}^{d} C_j x_{r-j-k}(z) \qquad \text{(using hypothesis i))} \\ &= \sum_{k=0}^{r} A_k y_{r-k}(z). \end{aligned}$$

From (18) and hypothesis ii) we deduce that

$$\hat{C}\Phi^{r+1} z = y_{r+1}(z) \in \Omega \ , \ \forall z \in \hat{Z}_r.$$

So $\hat{Z}_r \subset \hat{Z}_{r+1}$ or equivalently $\hat{Z}_r = \hat{Z}_{r+1}$. ■

Second case, $\operatorname{Dim}\Omega = q < n$.
Since every C_i is a $q \times n$ matrix, we define $\bar{C}_i, \bar{C}$ and $\bar{\Omega}$ by

$$\begin{aligned} \bar{C}_i &= \begin{pmatrix} C_i \\ 0 \end{pmatrix} \text{ is an } n \times n \text{ matrix} \\ \bar{C} &= (\bar{C}_0, \bar{C}_1, \ldots, \bar{C}_d, \underbrace{0, \ldots, 0}_{(r-d)-times}) \\ \bar{\Omega} &= \Omega \times \{0_{I\!R^{n-d}}\} \subset I\!R^n. \end{aligned}$$

Now consider the new observations $\bar{y}_i = \bar{C} e_i$ and $\bar{y}_i^p = \bar{C} e_i^p$, where $(e_i)_i$ and $(e_i^p)_i$ are respectively described by (14), (15). Clearly we have

$$y_i^p \in \Omega \iff \bar{y}_i^p \in \bar{\Omega}.$$

Thus the set E_d is described by

$$E_d = \{\Theta_d \ / \ \bar{y}_i^p \in \bar{\Omega} \ , \ \forall i \geqslant 0\}.$$

Since $\bar{y}_i^p = \sum_{j=0}^{d} \bar{C}_j x_{i-j}^p$, where $\bar{C}_i$ are matrices of order n, and $\mathrm{Dim}\,\bar{\Omega} = n$, we apply the results of the first case a) to deduce the following proposition.

Proposition 8 *Suppose the following hypothesis to hold*
i) $\bar{C}_i$ *commutes with* A_j *for every* $i \in \{0, \ldots, d\}$ *and every* $j \in \{0, \ldots, r\}$.
ii) $\sum_{i=0}^{r} A_i \bar{\Omega} \subset \Omega$.
Then the set E_d *is characterized by* (17).

4.2. Application

Consider the system

$$\begin{cases} x_{i+1} = \sum_{j=0}^{r} A_j x_{i-j} \ , \ i \geqslant 0 \\ x_0 \quad \text{given} \\ x_k = \alpha_k \ , \ -r \leqslant k \leqslant -1 \end{cases} \tag{19}$$

with the output function

$$y_i = x_i \ , \ i \geqslant 0$$

and the constraints

$$\|y_i\| \leqslant \epsilon \ , \ \forall i \geqslant 0$$

where $\epsilon > 0$ is a fixed integer. Then we have the following result.

Proposition 9 *If* $\sum_{i=0}^{r} \|A_i\| \leqslant 1$ *then the set of all admissible perturbations is characterized by* $E_d = \{\Theta_d \ / \ \|x_i^p\| \leqslant \epsilon \ , \ \forall i \in \{0, \ldots, I+2+r\}\}$ *where* $(x_i^p)_i$ *is the perturbed trajectory of system* (19).

Proof.
We show that the hypothesis of proposition 7 are verified. Indeed, for every $w_i \in \Omega = B(0, \epsilon), i = 0, \ldots, r$, we have

$$\|\sum_{i=0}^{r} A_i w_i\| \leqslant \sum_{i=0}^{r} \|A_i\| \|w_i\| \leqslant \epsilon.$$

Hence $\sum_{i=0}^{r} A_i \Omega \subset \Omega$. ∎

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