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Stability analysis of nonlinear control systems with unconstrained fuzzy predictive controllers

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Stability analysis of nonlinear control systems with unconstrained fuzzy predictive controllers using input–output plant models (e.g. DMC, GPC) and nonlinear plants with delays is discussed in the paper. The idea is precisely explained using an example of control systems with fuzzy DMC (FDMC) controllers. The considered FDMC controller is based on analytical formulation of the DMC predictive control algorithm and Takagi–Sugeno fuzzy modeling. The stability analysis of the closed–loop nonlinear control system is based on a transformation of its description to the appropriate state-space form. Then the Tanaka–Sugeno stability criterion can be applied, consisting in solving a set of Lyapunov–type inequalities. The design procedure including the stability analysis is illustrated on examples.

Key words: model predictive control, nonlinear systems, fuzzy systems, stability

1. Introduction

Stability analysis of control systems with model–based predictive fuzzy controllers based on an analytical formulation is the subject of the paper. The analysis is formulated in a general way, thus applying to a wide range of nonlinear control systems described by Takagi–Sugeno fuzzy models, the FDMC (Fuzzy DMC) and FGPC (Fuzzy GPC) control systems being special cases of interest. The idea is to transform the control system description to the form which enables application of the stability criterion formulated by Tanaka and Sugeno [15].

There are several papers where the mentioned stability criterion is used for control system analysis. Most of them consider control systems with state–feedback controllers [15–18]. In [14] the case with state–feedback controllers with observers is considered. The paper [8] discusses many aspects of stability of systems described by

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Takagi–Sugeno models. In [7] stability of control systems with dynamic output controllers is investigated.

Organization of the paper is as follows. First, fuzzy models of Takagi–Sugeno type and the original version of the Tanaka–Sugeno stability criterion are reminded. Then the considered analytical predictive controllers are presented. Next section contains development of the control system description that makes application of the Tanaka–Sugeno stability criterion possible. In section 5 two application examples of the proposed methodology are presented. A summary concludes the paper.

2. Stability criterion for Takagi–Sugeno fuzzy systems

Systems described by the Takagi–Sugeno fuzzy models (TS models) [13] are considered in this section. The model consists of the following l rules:

Ri: if $x(k)$ is F_1^i and ... and $x(k-n+1)$ is F_n^i, then

$$x^i(k+1) = a_1^i \cdot x(k) + \ldots + a_n^i \cdot x(k-n+1), \tag{1}$$

where $x(k), \ldots, x(k-n+1)$ – state variables, $F_1^i, \ldots, F_n^i$– fuzzy sets, $i = 1, \ldots, l$, l– number of rules.

The i–th local model (1) can be written as:

$$\mathbf{x}^i(k+1) = \mathbf{A}_i \cdot \mathbf{x}(k), \tag{2}$$

where

$$\mathbf{A}_i = \begin{bmatrix} a_1^i & \ldots & a_{n-1}^i & a_n^i \\ 1 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & 1 & 0 \end{bmatrix}, \quad \mathbf{x}(k) = \begin{bmatrix} x(k) \\ x(k-1) \\ x(k-2) \\ \vdots \\ x(k-n+2) \\ x(k-n+1) \end{bmatrix}.$$

The output of the TS fuzzy model is given by:

$$\mathbf{x}(k+1) = \frac{\sum_{i=1}^{l} w_i(k) \cdot \mathbf{A}_i \cdot \mathbf{x}(k)}{\sum_{i=1}^{l} w_i(k)}, \tag{3}$$

where $w_i(k) = \prod_{j=1}^{n} F_j^i\left(x(k-j+1)\right)$ is the i–th rule activation level (firing strength) at time instant k. The activation levels $w_i(k)$ will be further denoted by w_i, to shorten the description.

Tanaka and Sugeno formulated the following sufficient stability condition, arising from the direct Lyapunov method [15].

Theorem 1: *The fuzzy system (2) is asymptotically stable if there is a positive definite matrix* $\mathbf{P}$ *such that for all matrices* $\mathbf{A}_i$ *the following inequalities are fulfilled:*

$$\mathbf{A}_i^T \cdot \mathbf{P} \cdot \mathbf{A}_i - \mathbf{P} < 0, i = 1, \ldots, l. \quad (4)$$

To find the matrix $\mathbf{P}$ one can solve the set of linear matrix inequalities (4) [14, 17], e.g. using the Matlab LMI toolbox. However, it is worth to check first if the following necessary conditions for the matrix $\mathbf{P}$ to exist are satisfied:

Condition 1: The matrix $\mathbf{P}$ does not exist if any of the local systems is unstable;

Condition 2: If $\mathbf{A}_i$ matrices are stable and nonsingular, then $\mathbf{A}_i \cdot \mathbf{A}_j$ are stable matrices if the matrix $\mathbf{P}$ exists. Therefore, if there is an unstable matrix $\mathbf{A}_i \cdot \mathbf{A}_j$, then the matrix $\mathbf{P}$ does not exist [15].

3. Fuzzy predictive controllers

Fuzzy predictive controllers designed using input–output process models are considered in the paper, in particular Fuzzy DMC (FDMC) [10] and Fuzzy GPC (FGPC) algorithms. The idea of the predictive algorithms is to anticipate control system behavior many steps ahead during calculation of the control signal, see e.g. [1]. Future output values are derived from history of the control system and from the knowledge about future conditions of its work. The control signal changes are calculated in such a way that future behavior of the control system satisfies assumed criteria, usually minimization of a performance function based on output errors and control increments.

The design procedure will be explained on example of the FDMC controller. There are certain variants of FDMC algorithms, the reader is referred to [10]. This paper concerns the FDMC controller based on analytical formulation, i.e. when the control law is calculated in an analytical way, without taking into account constraints on controls or outputs. It is possible in this case to perform stability analysis of the resulting nonlinear closed–loop control system, as it will be shown in this paper. It should be mentioned that it is recommended to design and analyze first predictive controllers without constraints even in constrained cases, because it is then easier to analyze its basic properties and tune the parameters, see e.g. [12]. In constrained cases, the DMC algorithm is applied using constrained optimization at each step, or using the analytical (unconstrained) controller in a suitable anti–wind–up structure (especially in SISO cases).

3.1. Linear DMC algorithm based on analytical formulation

The DMC algorithm aims at minimization of the performance function:

$$J = \sum_{i=1}^{p} \left(y^{sp}(k) - y(k+i|k)\right)^2 + \lambda \cdot \sum_{i=0}^{s} \left(\Delta u(k+i|k)\right)^2, \quad (5)$$

where p and s denote prediction and control horizons, respectively, $\lambda \geqslant 0$ is a penalty coefficient, $y^{sp}(k)$ is a set–point value (which is assumed constant over the prediction horizon), $\Delta u(k+i|k) = u(k+i|k) - u(k+i-1|k)$ is a change in the manipulated variable for $(k+i)$–th time step calculated at k–th time step and $y(k+i|k)$ is an output value for $(k+i)$–th time step predicted at k–th time step. The predicted output signal can be decomposed into the following components:

$$y(k+i|k) = y(k) + v(k+i|k) + \Delta y(k+i|k), \tag{6}$$

where $y(k)$ is the measured output value, $v(k+i|k)$ depends on past control increments only and $\Delta y(k+i|k)$ depends on current and future control increments, $i = 0, \ldots, p$.

The DMC algorithm uses a control plant model in the form of the step response. So it can be written that

$$\mathbf{\Delta y}(k) = \mathbf{G} \cdot \mathbf{\Delta u}(k), \tag{7}$$

where
$$\mathbf{\Delta y}(k) = \left[\begin{array}{ccc} \Delta y(k+1|k) & \ldots & \Delta y(k+p|k) \end{array} \right]^T,$$
$$\mathbf{\Delta u}(k) = \left[\begin{array}{ccc} \Delta u(k|k) & \ldots & \Delta u(k+s-1|k) \end{array} \right]^T,$$

$\mathbf{G}$ is usually called a dynamic matrix, it is composed of elements of the plant step response [1, 6, 9]. The optimization problem (5) has a unique solution which can be expressed as

$$\mathbf{\Delta u}(k) = (\mathbf{G}^T \cdot \mathbf{G} + \lambda \cdot \mathbf{I})^{-1} \cdot \mathbf{G}^T \cdot (\mathbf{e}(k) - \mathbf{v}(k)), \tag{8}$$

where
$$\mathbf{e}(k) = \left[\begin{array}{ccc} y^{sp}(k) - y(k) & \ldots & y^{sp}(k) - y(k) \end{array} \right]^T \in \mathrm{Re}^p,$$
$$\mathbf{v}(k) = \left[\begin{array}{ccc} v(k+1|k) & \ldots & v(k+p|k) \end{array} \right]^T$$

and **I** denotes the identity matrix. The inverse matrix in (8) usually exists, possible singularity can be avoided by setting appropriate value of $\lambda > 0$ or by using shortened control horizon ($s < p$), see e.g. [1] for more information.

Only the first element of the vector $\mathbf{\Delta u}(k)$ is applied to the process and the procedure is repeated at each next time step. Therefore, after simple algebraic manipulations, the following control law structure can be derived from (8):

$$\Delta u(k) = r_0 \cdot e(k) + \sum_{j=1}^{p-1} r_j \cdot \Delta u(k-j), \tag{9}$$

where $u(k)$ – manipulated variable at the k–th time step, $\Delta u(k-j)$ – past changes in the manipulated variable, $e(k) = y^{sp}(k) - y(k)$ – control error at the k–th time step, r_j – resulting coefficients of the controller [9]. The structure of the controller given by (9) is shown in Fig. 1.

It should be mentioned that the number p in the upper limit of the sum in (9) is equal to the number of step response coefficients taken into account in the formulation of the DMC controller. We assume in this paper that the prediction horizon is equal to p (a very reasonable assumption), therefore the same symbol p in (5) and (9).

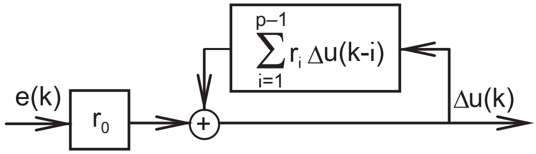

Figure 1. Block diagram of the DMC controller described by equation (9)

3.2. Fuzzy DMC algorithm based on analytical formulation

Application of a linear DMC algorithm for a nonlinear plant can bring poor results. Performance of the control system can be improved by using a nonlinear controller, e.g. a FDMC controller. It is based on Takagi–Sugeno (TS) fuzzy modeling [13] and the DMC algorithm described in the previous section.

The idea of the FDMC controller is to obtain first, for every rule in the TS model (corresponding to a local region in the input–output space), a local linear process model in the form of a vector of step response coefficients. Then every such vector is used to design a local DMC controller with coefficients r_j^i, where superscript i denotes the number of the rule (region):

$$\Delta u^i(k) = r_0^i \cdot e(k) + \sum_{j=1}^{p-1} r_j^i \cdot \Delta u(k-j). \tag{10}$$

In every FDMC controller iteration (time step) the weights for local controllers, i.e. activation levels of fuzzy rules depending on the process inputs and outputs, are derived in a fuzzy way. Then output of the controller is calculated summing up weighted local controller outputs:

$$\Delta u(k) = \frac{\sum\limits_{i=1}^{l} w_i \cdot \Delta u^i(k)}{\sum\limits_{i=1}^{l} w_i}, \tag{11}$$

where $\Delta u^i(k)$ – outputs of local controllers given by (10), w_i – weights (activation levels of fuzzy rules), $i = 1, \ldots, l$, l – number of rules. Block diagram of the FDMC controller is shown in Fig. 2, where $\widetilde{w}_i = w_i / \sum_{i=1}^{l} w_i$ – normalized weights.

3.3. Fuzzy GPC algorithm

FGPC (Fuzzy GPC) controller can be constructed using analogous design procedure as it was in the FDMC case. This time the following local plant models are used:

$$y^i(k+1) = \sum_{jb=1}^{n_B} b_{jb}^i \cdot y(k-jb+1) + \sum_{jc=1}^{m_C} c_{jc}^i \cdot u(k-jc+1), \tag{12}$$

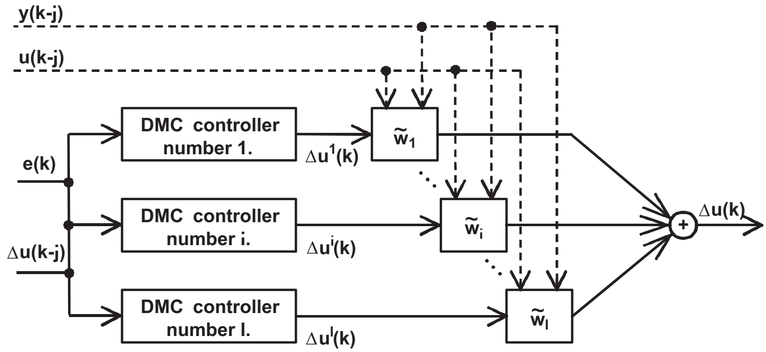

Figure 2. Block diagram of the FDMC controller

where b^i_{jb}, c^i_{jc} – local model coefficients, $jb = 1, \ldots, n_B$, $jc = 1, \ldots, m_C$. After suitable manipulations, analogous to that from section 3.1, the following local control laws are obtained (see e.g. [1,5]):

$$\Delta u^i(k) = (r_e)^i \cdot e(k) + \sum_{j=1}^{m_C-1} (r_u)^i_j \cdot \Delta u(k-j) + \sum_{j=1}^{n_B+1} (r_y)^i_j \cdot y(k-j+1), \quad (13)$$

where $(r_e)^i$, $(r_u)^i_j$, $(r_y)^i_j$ – coefficients obtained after the transformation.

It is worth to notice that the GPC control law is structured similarly to the DMC control law, but instead of factors dependent on past input increments only, factors dependent on past outputs are also present. Moreover, the number of factors dependent on past input increments is smaller. The output of the FGPC controller is calculated as in the case of the FDMC controller, see (11).

4. Stability analysis

This section describes derivation of the considered control system description into the form that enables to apply the stability criterion given in section 2. Two ways to perform this derivation are possible. The first one is more readable and the second one will be only shortly commented.

It is assumed that both the controller and the plant are described by the fuzzy TS models. The general case is examined and the controlled plant with time delay d is assumed. The plant model consists of the following loy rules:

Oi: if $y(k)$ is Y^i_1 and ... and $y(k-n_P+1)$ is $Y^i_{n_P}$ and
$u(k)$ is U^i_1 and ... and $u(k-m_P+1)$ is $U^i_{m_P}$ then

$$y^i(k+1) = b^i_1 \cdot y(k) + \ldots + b^i_{n_B} \cdot y(k-n_B+1) + \\ + c^i_1 \cdot u(k-d) + \ldots + c^i_{m_C} \cdot u(k-d-m_C+1), \quad (14)$$

where $b_1^i, \ldots, b_{n_B}^i, c_1^i, \ldots, c_{m_C}^i$ – i–th local model coefficients, $y(k)$ – the output, $u(k)$ – the manipulated variable. The output of the plant model is given by the standard equation:

$$y(k+1) = \frac{\sum\limits_{i=1}^{loy} w_i \cdot y^i(k+1)}{\sum\limits_{i=1}^{loy} w_i}, \tag{15}$$

where the weights $w_i = \prod\limits_{h=1}^{n_P} Y_h^i \left(y(k-h+1)\right) \cdot \prod\limits_{h=1}^{m_P} U_h^i \left(u(k-h+1)\right)$ are defined by activation levels (firing strength) of the rules, compare (3).

The controller is described by the following model:

Pj: if $y(k)$ is $(Y_c)_1^j$ and … and $y(k-n_R+1)$ is $(Y_c)_{n_R}^j$ and
$u(k-1)$ is $(U_c)_2^j$ and… and $u(k-m_R+1)$ is $(U_c)_{m_R}^j$ then

$$u^j(k) = f_2^j \cdot u(k-1) + \ldots + f_{m_F}^j \cdot u(k-m_F+1) + \\ + g_1^j \cdot e(k) + \ldots + g_{n_G}^j \cdot e(k-n_G+1), \tag{16}$$

where $f_2^j, \ldots, f_{m_F}^j, g_1^j, \ldots, g_{n_G}^j$ – j–th local controller coefficients. The controller output is as follows:

$$u(k) = \frac{\sum\limits_{j=1}^{lou} w_j \cdot u^j(k)}{\sum\limits_{j=1}^{lou} w_j}, \tag{17}$$

where $w_j = \prod\limits_{h=1}^{n_R} (Y_c)_h^j \left(y(k-h+1)\right) \cdot \prod\limits_{h=2}^{m_R} (U_c)_h^j \left(u(k-h+1)\right)$.

In order to simplify the formulae resulting from algebraic manipulations, we will assume that

$$n = \max\left\{n_B, n_G + d\right\}, \tag{18}$$

$$m = \max\left\{m_C, m_F\right\}, \tag{19}$$

and lacking coefficients b_h^i (if $n_G + d > n_B$) or g_h^j (if $n_G + d < n_B$) in the plant model, as well as lacking coefficients c_h^i (if $m_F > m_C$) or f_h^j (if $m_F < m_C$) in the controller are assumed to be zero.

Let us introduce a quasi–state vector of the following form [4]:

$$\mathbf{x}_b(k) = \left[\; y(k) \;\ldots\; y(k-n+1) \quad u(k-d-1) \;\ldots\; u(k-d-m+1) \;\right]^T. \tag{20}$$

Then, the plant model will have the following form:

$$\mathbf{x}_b(k+1) = \sum_{i=1}^{loy} \widetilde{w}_i \cdot \mathbf{A}^i \cdot \mathbf{x}_b(k) + \sum_{i=1}^{loy} \widetilde{w}_i \cdot \mathbf{B}^i \cdot u(k-d), \tag{21}$$

$$y(k+1) = \sum_{i=1}^{loy} \widetilde{w}_i \cdot \mathbf{C}^i \cdot \mathbf{x}_b(k+1),$$

where $\widetilde{w}_i = w_i / \sum_{i=1}^{l} w_i$ and the matrices are as follows:

$$\mathbf{A}^i = \begin{bmatrix} b_1^i & b_2^i & \dots & b_{n-1}^i & b_n^i & c_2^i & c_3^i & \dots & c_{m-1}^i & c_m^i \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \ \mathbf{B}^i = \begin{bmatrix} c_1^i \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\mathbf{C}^i = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}. \tag{22}$$

The controller output at the $(k-d)$–th step is described by the equation:

$$u(k-d) = \sum_{j=1}^{lou} \widetilde{w}_j \cdot \mathbf{F}^j \cdot \mathbf{x}_b(k) + \sum_{j=1}^{lou} \widetilde{w}_j \cdot \mathbf{G}^j \cdot \mathbf{y}^{sp}(k), \tag{23}$$

where

$$\mathbf{F}^j = \begin{bmatrix} 0 & 0 & \dots & 0 & -g_1^j & \dots & -g_{n-d-1}^j & -g_{n-d}^j & f_2^j & f_3^j & \dots & f_{m-1}^j & f_m^j \end{bmatrix},$$

$\widetilde{w}_j = w_j / \sum_{j=1}^{l} w_j$ and $\mathbf{y}^{sp}(k)$ is the external input vector. Its form and the form of the matrix $\mathbf{G}^j$ arise from the applied analysis and are not listed here, because they do not influence the stability condition.

Inserting (23) into (21) results in the final form of the closed loop system equations:

$$\begin{aligned} \mathbf{x}_b(k+1) &= \mathbf{A} \cdot \mathbf{x}_b(k) + \mathbf{B} \cdot \mathbf{y}^{sp}(k), \\ y(k+1) &= \mathbf{C} \cdot \mathbf{x}_b(k+1), \end{aligned} \tag{24}$$

where $\mathbf{A} = \sum_{i=1}^{loy} \widetilde{w}_i \cdot \sum_{j=1}^{lou} \widetilde{w}_j \cdot \mathbf{A}_{ij}$, $\mathbf{B} = \sum_{i=1}^{loy} \widetilde{w}_i \cdot \sum_{j=1}^{lou} \widetilde{w}_j \cdot \mathbf{B}^i \cdot \mathbf{G}^j$, $\mathbf{C} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$ and the matrices $\mathbf{A}_{ij} = \mathbf{A}^i + \mathbf{B}^i \cdot \mathbf{F}^j$ have the form:

$$\mathbf{A}_{ij} = \begin{bmatrix} bg_1^{i,j} & bg_2^{i,j} & \dots & bg_d^{i,j} & bg_{d+1}^{i,j} & \dots & bg_{n-1}^{i,j} & bg_n^{i,j} & cf_2^{i,j} & cf_3^{i,j} & \dots & cf_{m-1}^{i,j} & cf_m^{i,j} \\ 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -g_1^j & \dots & -g_{n-d-1}^j & -g_{n-d}^j & f_2^j & f_3^j & \dots & f_{m-1}^j & f_m^j \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \tag{25}$$

$$\text{where} \quad bg_{in}^{i,j} = \begin{cases} b_{in}^i; & in < d+1 \\ b_{in}^i - c_1^i \cdot g_{in-d}^j; & in \geqslant d+1 \end{cases}, \; in = 1, \dots, n;$$

$$cf_{im}^{i,j} = c_{im}^i + c_1^i \cdot f_{im}^j, \qquad im = 2, \dots, m.$$

There is another way the considered control system description can be transformed to the demanded form, analogous to that presented in [7]. It is only sketched below.

First, from (14) and (15) and using assumptions (18) and (19) one can obtain:

$$y(k+1) = \sum_{i=1}^{loy} \widetilde{w}_i \cdot \left(\sum_{in=1}^{n} b_{in}^i \cdot y(k - in + 1) + \sum_{im=1}^{m} c_{im}^i \cdot u(k - d - im + 1) \right) \tag{26}$$

and, after using (16) and (17),

$$u(k-d) = \sum_{j=1}^{lou} \widetilde{w}_j \cdot \sum_{io=2}^{m} f_{io}^j \cdot u(k - d - io + 1) +$$
$$+ \sum_{j=1}^{lou} \widetilde{w}_j \cdot \left(\sum_{ip=1}^{n-d} g_{ip}^j \cdot y^{sp}(k-d-ip+1) - \sum_{ip=1}^{n-d} g_{ip}^j \cdot y(k-d-ip+1) \right). \tag{27}$$

Then, after substituting (26) into (27) and certain manipulations we get the formula:

$$y(k+1)=\sum_{i=1}^{loy}\widetilde{w}_i\cdot\sum_{j=1}^{lou}\widetilde{w}_j\cdot\left(\sum_{in=1}^{n}bg_{in}^{i,j}\cdot y(k-in+1)+\sum_{im=2}^{m}cf_{im}^{i,j}\cdot u(k-d-im+1)\right)+$$
$$+\sum_{i=1}^{loy}\widetilde{w}_i\cdot\sum_{j=1}^{lou}\widetilde{w}_j\cdot\sum_{ip=1}^{n-d}c_1^i\cdot g_{ip}^j\cdot y^{sp}(k-d-ip+1). \tag{28}$$

It is sufficient now to introduce the quasi–state vector (20), then after suitable manipulations the same state space description (24) with local system matrices (25) is obtained. It is now also possible to use the stability criterion given in section 2. To prove stability of the whole control system it is enough to find a positive definite matrix **P** such that matrices $\mathbf{A}_{ij}^T\cdot\mathbf{P}\cdot\mathbf{A}_{ij}-\mathbf{P}$ are negative definite for all $i=1,\ldots,loy$, $j=1,\ldots,lou$.

Remark 1: If membership functions are the same in the controller and in the control plant, then it is possible (due to equalities $w_i\cdot w_j=w_j\cdot w_i$) to decrease number of rules in the closed loop system; see [15]. The system matrix **A** from (24) can then be further transformed into the form:

$$\mathbf{A}=\sum_{i=1}^{l}\widetilde{w}_i\cdot\widetilde{w}_i\cdot\mathbf{A}_{ii}+\sum_{i=1}^{l}\sum_{j=i+1}^{l}2\cdot\widetilde{w}_i\cdot\widetilde{w}_j\cdot\mathbf{A}_{ij*}, \tag{29}$$

where $\mathbf{A}_{ij*}=\frac{\mathbf{A}_{ij}+\mathbf{A}_{ji}}{2}$.

Remark 2: In [14] slightly less conservative criterion was presented, but under the same assumption as in the previous remark, that membership functions in the plant model and in the controller are the same. The theorem is as follows:

Theorem 2: *If membership functions in the controller and in the control plant model are the same and the number of rules active in every time step is not bigger then s, where $1<s<l$, then fuzzy system is stable if there are: a common positive definite matrix* $\mathbf{P}$ *and a common positive semidefinite matrix* $\mathbf{Q}$ *such that the following inequalities are fulfilled:*

$$\mathbf{A}_{ii}^T\cdot\mathbf{P}\cdot\mathbf{A}_{ii}-\mathbf{P}+(s-1)\cdot\mathbf{Q}<0 \text{ for } i=1,\ldots,l, \tag{30}$$
$$\mathbf{A}_{ij*}^T\cdot\mathbf{P}\cdot\mathbf{A}_{ij*}-\mathbf{P}-\mathbf{Q}\leqslant 0 \text{ for } i<j \tag{31}$$

except i and j such, that $w_i\cdot w_j=0\;\forall k$.

The presented considerations will now be applied to the control system with the FDMC controller and the plant described by Takagi–Sugeno models. The control plant with time delay is described by the following model:

Oi: if $y(k)$ is Y_1^i and ... and $y(k-n_P+1)$ is $Y_{n_P}^i$ and
$u(k)$ is U_1^i and ... and $u(k-m_P+1)$ is $U_{m_P}^i$ then

$$y^i(k+1)=b_1^i\cdot y(k)+\ldots+b_n^i\cdot y(k-n+1)+$$
$$+c_1^i\cdot u(k-d)+\ldots+c_m^i\cdot u(k-d-m+1). \tag{32}$$

The FDMC controller consists of several linear DMC controllers, as described in section 3. Therefore, every controller rule has the following form:

Pj: if $y(k)$ is $(Y_c)_1^j$ and ... and $y(k-n_R+1)$ is $(Y_c)_{n_R}^j$ and
$u(k-1)$ is $(U_c)_2^j$ and ... and $u(k-m_R+1)$ is $(U_c)_{m_R}^j$ then

$$u^j(k) = u(k-1) + r_0^j \cdot e(k) + r_1^j \cdot \Delta u(k-1) + \ldots + \\ + r_{p-1}^j \cdot \Delta u(k-p+1), \tag{33}$$

where (10) has been written with $\Delta u^i(k)$ replaced with $u^i(k) - u(k-1)$.

In order to obtain controller formulation dependent not on control changes but on control values, equation (33) is transformed to the form:

$$u^j(k) = r_0^j \cdot e(k) + (1 + r_1^j) \cdot u(k-1) + (r_2^j - r_1^j) \cdot u(k-2) + \ldots + \\ + (r_{p-1}^j - r_{p-2}^j) \cdot u(k-p+1) - r_{p-1}^j \cdot u(k-p). \tag{34}$$

In this case, assuming that $m < p+1$, coefficients of the controller (16) and matrices (25), resulting from the assumed models, are as follows:

$$g_{in}^j = \begin{cases} r_0^j; & in = 1 \\ 0; & in \neq 1 \end{cases},$$

$$f_{im}^j = \begin{cases} 1 + r_1^j; & im = 2 \\ r_{im-1}^j - r_{im-2}^j; & im > 2,\ im \leqslant p \\ -r_{p-1}^j; & im = p+1 \end{cases},$$

$$bg_{in}^{i,j} = \begin{cases} b_{in}^i - c_1^i \cdot r_0^j; & in = d+1 \\ b_{in}^i; & in \neq d+1 \end{cases},$$

$$cf_{im}^{i,j} = \begin{cases} c_{im}^i + c_1^i \cdot f_{im}^j; & im > 2,\ im \leqslant m \\ c_1^i \cdot f_{im}^j; & im > m \end{cases},$$

$in = 1, \ldots, n$, $im = 2, \ldots, p+1$. Finally, $cf_{im}^{i,j}$ parameters are as follows:

$$cf_{im}^{i,j} = \begin{cases} c_{im}^i + c_1^i \cdot \left(1 + r_1^j\right); & im = 2 \\ c_{im}^i + c_1^i \cdot \left(r_{im-1}^j - r_{im-2}^j\right); & im > 2,\ im \leqslant m \\ c_1^i \cdot \left(r_{im-1}^j - r_{im-2}^j\right); & im > m,\ im \leqslant p \\ -c_1^i \cdot r_{p-1}^j; & im = p+1 \end{cases}.$$

Substituting the obtained coefficients into equation (25) results in the following family of matrices $\mathbf{A}_{ij}$:

$$
\mathbf{A}_{ij}=\left[\begin{array}{ccccccccccccccccc}
b_1^i & b_2^i & \dots & b_d^i & b_{d+1}^i-c_1^i r_0^j & \dots & b_{n-1}^i & b_n^i & c_2^i+c_1^i\left(1+r_1^j\right) & c_3^i+c_1^i\left(r_2^j-r_1^j\right) & \dots & \dots & c_m^i+c_1^i\left(r_{m-1}^j-r_{m-2}^j\right) & c_1^i\left(r_m^j-r_{m-1}^j\right) & \dots & c_1^i\left(r_{p-1}^j-r_{p-2}^j\right) & -c_1^i r_{p-1}^j \\
1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & \dots & 0 & 0 \\
0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & -r_0^j & \dots & 0 & 0 & 1+r_1^j & r_2^j-r_1^j & \dots & \dots & r_{m-1}^j-r_{m-2}^j & r_m^j-r_{m-1}^j & \dots & r_{p-1}^j-r_{p-2}^j & c_1^i r_{p-1}^j \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & \dots & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & \dots & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & \dots & 1 & 0
\end{array}\right]. \tag{35}
$$

It was assumed that $m \leqslant p+1$ in the above considerations because the number of terms depending on past outputs in the model (32) should not exceed the number of elements in the step response (equal to p). However, the matrices $\mathbf{A}_{ij}$ can be derived also in the opposite case.

The stability analysis can be analogously applied to the control system with the FGPC controller. But the general form of local GPC controllers (13) does not fit, at the first glance, to the controller form (16), where there is not explicit dependence on past plant outputs. However, the stability conditions presented in this section do not depend on values of the set–points $y^{sp}(k)$. Therefore, without loss of generality we can assume $y^{sp}(k) = y^{sp}(k+1) = \ldots = 0$ and in this way (16) covers also the case of the GPC control law. The detailed form of matrices $\mathbf{A}_{ij}$ can be easily obtained from (25).

5. Simulation examples

In this section applications of the presented stability analysis will be demonstrated on exemplary control systems with the FDMC controllers.

Example 1

Let the control plant be described by the following model:

Rule 1: if $y(k)$ is Z_1, then $y^1(k+1) = 0.7 \cdot y(k) + 0.8 \cdot u(k)$;

Rule 2: if $y(k)$ is Z_2, then $y^2(k+1) = 0.3 \cdot y(k) + 0.2 \cdot u(k)$;

with membership functions shown in Fig. 4. Step responses of the local models are presented in Fig. 3.

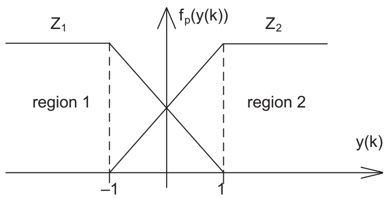

Figure 3. Step responses of local models

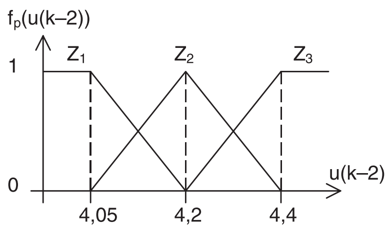

Figure 4. Membership functions of exemplary control plant

First, it has been tried to design a linear (non-fuzzy) DMC controller for the presented plant, extended by a first order discrete filter (with a pole in 0.6) on the set–point input. But there were no satisfactory results. If the controller was designed to work well for negative set–points (region 1), then it worked too slowly for positive set–point values (Fig. 6). If it was designed to operate well for positive set–points, then unsatisfactory behavior was obtained for negative set–points (Fig. 5). Satisfactory control system operation was obtained with a fuzzy DMC controller (Fig. 7, 8). Coefficients of two local controllers are given in Table 1. It was assumed that membership functions in the plant model and in the controller are the same. Because of that the considered closed–loop system consists of three rules.

Table 1. Local controllers' coefficients in example 1; $\lambda = 0.1$

i	0	1	2	3	4	5	6
r_i^1	0.9999	−0.6300	−0.4410	−0.3087	−0.2161	−0.1512	−0.1060
r_i^2	2.1857	−0.1567	−0.0470	−0.0141	−0.0042	−0.0013	−0.0004

i	7	8	9	10	11	12
r_i^1	−0.0741	−0.0519	−0.0362	−0.0254	−0.0179	−0.0124
r_i^2	0	0	0	0	0	0

i	13	14	15	16	17	18
r_i^1	−0.0088	−0.0061	−0.0043	−0.0030	−0.0021	−0.0013
r_i^2	0	0	0	0	0	0

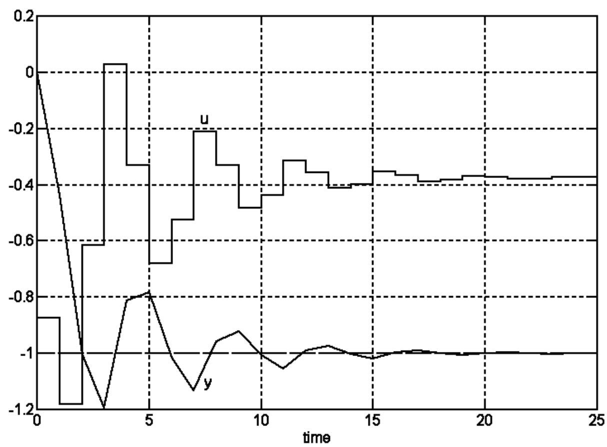

Figure 5. Response of control system with conventional DMC controller designed for region 2; set–point change from 0 to -1

After substituting controller coefficients into (35) three $\mathbf{A}_{ij}$ matrices are obtained. It is easy to check that these matrices are stable. Moreover, a positive definite matrix $\mathbf{P}$ satisfying the stability criterion was found using LMI Matlab toolbox. Therefore, the considered system is asymptotically stable. The matrices $\mathbf{A}_{ij}$ and $\mathbf{P}$ are not presented here because of their large dimensions, but they may be easily obtained using presented procedure.

In order to investigate conservativeness of the stability criterion the following experiment was done. Gain of the control plant was being increased (multiplied by k), with the controller as in the nominal case. The stability of the disturbed closed–loop system was checked after every change in k. The matrix $\mathbf{P}$ was found for $k \leqslant 2.372$. One of the $\mathbf{A}_{ij}$ matrices became unstable for $k = 2.395$. For $2.372 < k < 2.395$ the criterion does not determine the stability of the considered control system. Therefore, the stability

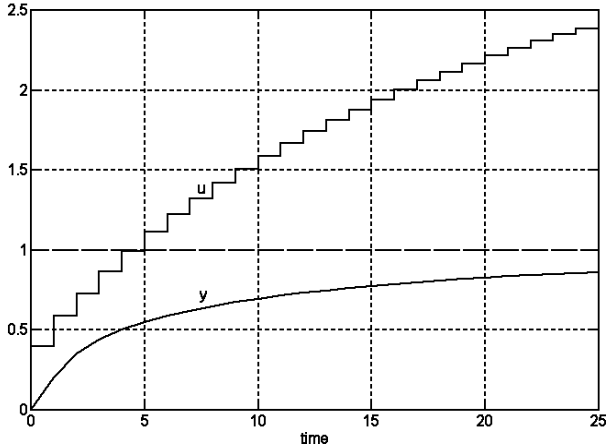

Figure 6. Response of control system with conventional DMC controller designed for region 1; set–point change from 0 to 1

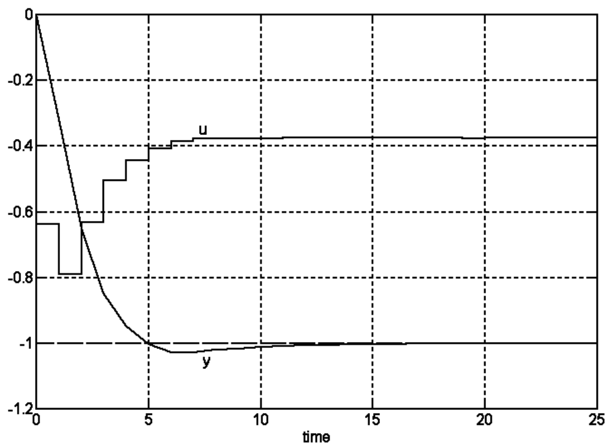

Figure 7. Response of control system with fuzzy DMC controller; set–point change from 0 to -1

criterion (providing sufficient conditions only) occurred to be not too conservative in this case.

Example 2

The control plant is an ethylene distillation column. It is a highly nonlinear plant with a large time delay. A control system for this plant was discussed in [10], together with presentation of sample simulation results. It was assumed that the model has the Hammerstein structure, i.e. it consists of a nonlinear statics and a linear dynamics. Such structure implies the choice of membership functions. Structure of the model and its static characteristics are shown in Fig. 9 and Fig. 10, respectively, where y – output concentration in ppm, u – reflux to product ratio, x_f – feed concentration.

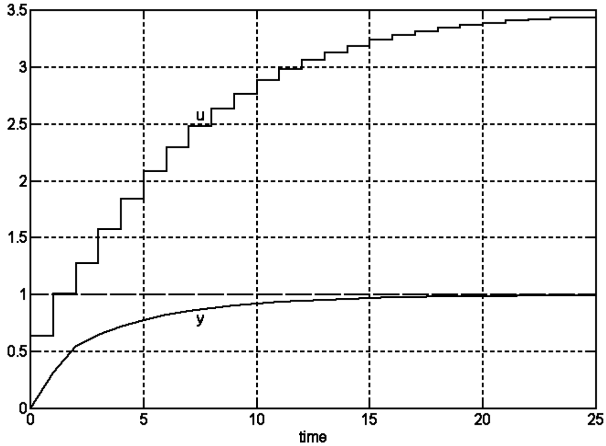

Figure 8. Response of control system with conventional Response of control system with fuzzy DMC controller; set–point change from 0 to 1

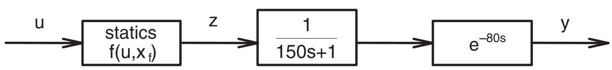

Figure 9. Block diagram of the control plant model; u – manipulated variable, x_f – measurable disturbance, y – output variable, z – output of the static part of the plant

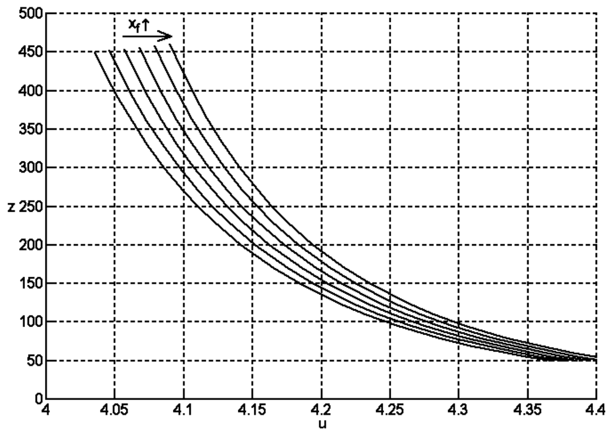

Figure 10. Static characteristics of the control plant

Now the stability of this control system will be checked. The control plant model for the sampling period $T_p = 40\ min$ is as follows:

Rule 1: if $u(k-2)$ is Z_1, then

$$y^1(k+1) = 0.7659 \cdot y(k) - 520.2638 \cdot u(k-2) + 2220.9067; \tag{36}$$

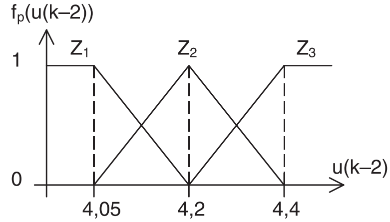

Figure 11. Membership functions of the control plant model

Rule 2: if $u(k-2)$ is Z_2, then

$$y^2(k+1) = 0.7659 \cdot y(k) - 253.5771 \cdot u(k-2) + 1102.4471;$$

Rule 3: if $u(k-2)$ is Z_3, then

$$y^3(k+1) = 0.7659 \cdot y(k) - 125.1030 \cdot u(k-2) + 563.8767;$$

with membership functions shown in Fig. 11.

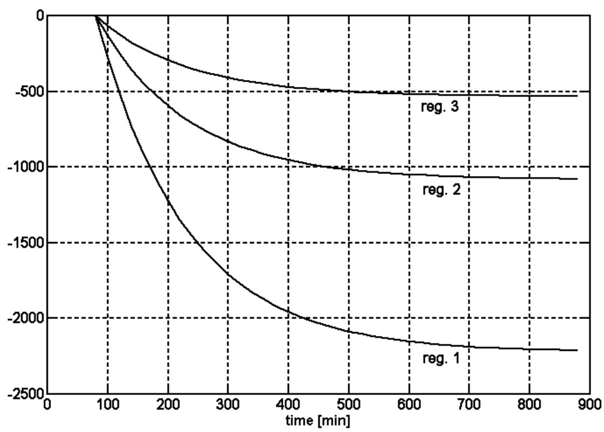

Figure 12. Normalized step responses of local models; regions of: 1 – large, 2 – medium and 3 – small impurity

Table 2. Coefficients of local controllers in example 2; $\lambda = 8e + 6$

i	0	1	2	3	4	5
r_i^1	–2.8433e–4	–4.2023e–1	–4.6978e–1	–3.5985e–1	–2.7565e–1	–2.1118e–1
r_i^2	–3.1038e–4	–2.5392e–1	–2.7318e–1	–2.0925e–1	–1.6027e–1	–1.2277e–1
r_i^3	–3.3222e–4	–1.4922e–1	–1.5584e–1	–1.1935e–1	–9.1400e–2	–6.9983e–2
i	6	7	8	9	10	11
r_i^1	–1.6177e–1	–1.2395e–1	–9.4981e–2	–7.2766e–2	–5.5775e–2	–4.2722e–2
r_i^2	–9.4050e–2	–7.2028e–2	–5.5177e–2	–4.2250e–2	–3.2345e–2	–2.4721e–2
r_i^3	–5.3572e–2	–4.0995e–2	–3.1355e–2	–2.3962e–2	–1.8291e–2	–1.3937e–2

i	12	13	14	15	16	17
r_i^1	–3.2728e–2	–2.5062e–2	–1.9138e–2	–1.4515e–2	–1.0921e–2	–8.0255e–3
r_i^2	–1.8878e–2	–1.4382e–2	–1.0901e–2	–8.1776e–3	–6.0656e–3	–4.4131e–3
r_i^3	–1.0590e–2	–8.0130e–3	–6.0241e–3	–4.4845e–3	–3.2882e–3	–2.3549e–3

i	18	19	20	21
r_i^1	–5.6905e–3	–3.7532e–3	–2.1609e–3	–9.3829e–4
r_i^2	–3.0647e–3	–1.9865e–3	–1.1484e–3	–4.9661e–4
r_i^3	–1.6242e–3	–1.0514e–3	–6.0451e–4	–2.6219e–4

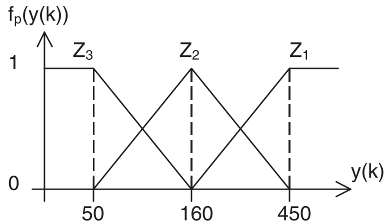

Figure 13. Membership functions of the controller

Then, using step responses of the control plant local models (Fig. 12) a FDMC controller was designed. The obtained coefficients of three local controllers are given in Table 2. Membership functions of the controller are shown in Fig. 13. Because membership functions of the controller are different than those of the control plant model, the considered closed–loop system consists of nine rules.

Now, parameters in the definition of the quasi–state vector (20) are: $d = 2$, $n = 3$ and $m = 22$. After substituting controller coefficients into (35) we obtain nine $\mathbf{A}_{ij}$ matrices. All $\mathbf{A}_{ij}$ matrices of the considered control system are stable. Using the LMI Matlab toolbox a positive definite matrix $\mathbf{P}$ satisfying the stability criterion was found for these matrices. The considered closed–loop system is therefore asymptotically stable. The matrices $\mathbf{A}_{ij}$ and $\mathbf{P}$ are not listed here in order to limit the paper size, but they may be easily obtained using the presented method.

In order to check how conservative the obtained result is, an experiment like that in example 1 was made, i.e. increasing the gain of the control plant by k with the controller as in the nominal case. The $\mathbf{P}$ matrix was found for $k \leqslant 1.020$. One of the $\mathbf{A}_{ij}$ matrices became unstable for $k = 1.336$. For $1.020 < k < 1.336$ the criterion does not determine the stability of the considered control system. Moreover, stability regions of all controller–plant combinations (resulting in $\mathbf{A}_{ij}$ matrices in the stability criterion) were calculated. Selected stability regions corresponding to matrices $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{13}$ (obtained for controller–plant combinations with the 1^{st} local plant model) are presented in Figs. 14–16. In the presented figures, the controller–plant combination is unstable, if the parameters are in the area above the curve; the area below the curve corresponds to stability (it should be reminded that stability of every controller–plant combination is only a necessary condition for the matrix $\mathbf{P}$ to exist).

In Figs. 14–16 relative values of the parameters are given, with respect to nominal values $(d_o)_{nom}$ and $(k_o)_{nom}$. The nominal values are marked in the picture by intersection of solid lines and additionally by an asterisk. Among the controller–plant combinations, the one corresponding to the case shown in Fig. 16 is the closest to the stability border.

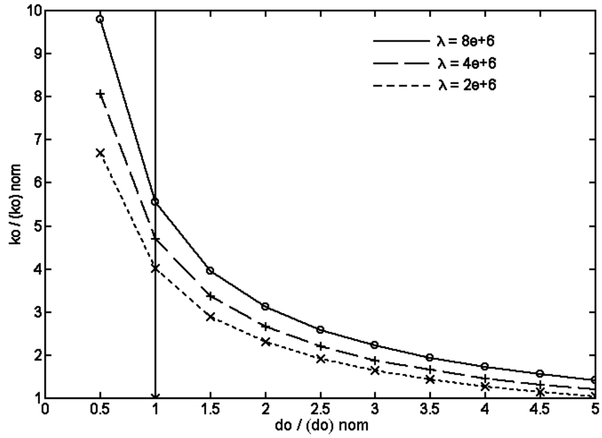

Figure 14. Stability regions of the controller–plant combinations consisting of the 1^{st} local plant model and the 1^{st} local controller.

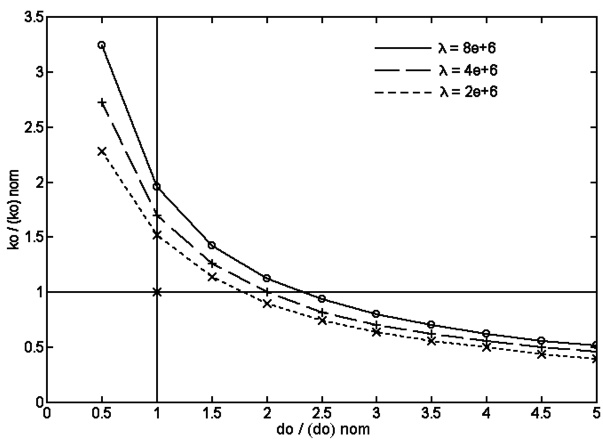

Figure 15. Stability regions of the controller–plant combinations consisting of the 1^{st} local plant model and the 2^{nd} local controller.

It was also checked, how changes in the controller parameter λ influence the control system stability. When decreasing λ, the stability regions of all controller–plant combi-

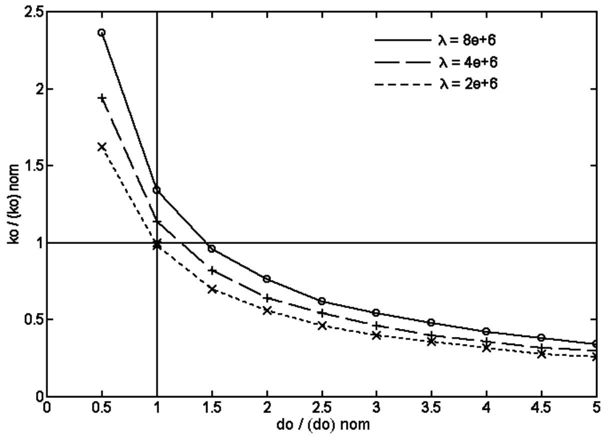

Figure 16. Stability regions of the controller–plant combinations consisting of the 1^{st} local plant model and the 3^{rd} local controller.

nations become smaller, as expected. For $\lambda = 4e + 6$ (two times smaller then originally assumed) and nominal values $(d_o)_{nom}$ and $(k_o)_{nom}$, all $\mathbf{A}_{ij}$ matrices are still stable, but the LMI solving procedure did not find the Lyapunov matrix $\mathbf{P}$. So, the criterion does not determine the stability of the considered control system. Further, when the parameter λ was assumed to be four times smaller ($\lambda = 2e + 6$), the controller–plant combination built of the 1^{st} local plant model and the 3^{rd} local controller occurred to be unstable (Fig. 16).

6. Conclusions

In the presented paper stability analysis of nonlinear control systems with fuzzy predictive controllers without constraints and based on input–output plant models is presented. The analysis is based on applying the Tanaka–Sugeno stability criterion to the appropriately reformulated control system description. Application of the method to two illustrative examples of control systems with FDMC controllers is presented showing its usefulness. It is worth to notice, that the formulation of the analysis method is general, so it can be used in the case of other similarly structured control systems.

Moreover, certain modifications of the Tanaka–Sugeno criterion and possibly other stability criteria can be used, because the considered control system was transformed to the standard form. In particular, if it can be assumed that limited number of local models is active at the same time sample and that the state variations are bounded then certain slightly relaxed stability conditions can also be applied for the considered control systems, see e.g. [3, 14]. A quite another criterion based on matrix norms, presented in [2], can be also used. But this criterion seems to be significantly more conservative than that presented in this paper and therefore less useful.

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