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End-point control of a flexible-link manipulator using state-dependent Riccati equation technique

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The problem of modeling and controlling the tip position of a one-link flexible manipulator is considered. The paper discusses the control strategy based on the nonlinear state Dependent Ricatti Equation (SDRE) design method in the context of application to robotics and manufacturing systems. Lagrangian Mechanics and the Assumed Mode Method have been used to derive a proposed dynamic model of a single-link flexible manipulator having a revolute joint. The model may be used in general to investigate the motion of the manipulator in the horizontal plane rest-to-rest rotational maneuver. The nonlinear SDRE control law is derived as minimizing a quadratic cost function that penalizes the states and the control input torques. Simulation results are presented for a single-link flexible manipulator to achieve a desired angular rotation of the link while simultaneously suppressing structural vibrations, and the effect of payload on the system response and vibration frequencies is investigated. The results are illustrated by a numerical example.

Key words: robotics, control of one-link flexible manipulator, nonlinear state-dependent Ricatti equation

1. Introduction

Flexible manipulator systems offer several advantages in contrast to the traditional rigid manipulator. These include faster response, lower energy consumption, requiring relatively smaller actuators, lower overall mass and, in general, lower overall cost [5]. However, due to its flexible nature the control of the flexible system is to take into account both the rigid body degree of freedom, and elastic degrees of freedom. It is important to recognize the flexible nature of the manipulator and construct a mathematical model for the system that accounts for the interactions with actuators and payload. The efficiency of a single-link flexible manipulator moving at high speed and having a payload is highly dependent on its dynamic behavior.

The Lagrangian Mechanics and the Assumed Mode Method have been used to drive a proposed dynamic model of a single-link flexible manipulator having a revolute joint. The link has been considered as an Euler-Bernoulli beam subjected to large angular

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displacement. To establish a model of a single-link flexible manipulator, the kinematics of a single link flexible manipulator is described, here, based on the equivalent rigid link system and a transformation matrix method. The overall motion of the flexible link manipulator consists of the rigid body motion, which is defined by the joint angle, and the elastic motion, which is defined by the first two modal coordinates. The application of Lagrangian equation yields two sets of equations. The first set is associated with the Rigid Body degree of freedom, and the other set is associated with the Elastic degrees of freedom. These two sets of equations of motion for a single-link flexible manipulator are nonlinear time varying and coupled second order ordinary differential equations. As a result, theory has emerged for design according to a number of methods, including feedback linearization [7], variable structure control [9], control Lyapunov functions [1], recursive backstepping and nonlinear H_∞ control [2]. In this paper we use the State Dependent Riccati Equation (SDRE) Technique for nonlinear control, which has recently appeared in the literature [3, 4, 10].

The main contribution of this paper is in adopting the SDRE approach to the needs of the flexible manipulator system and then proving its worthiness through tests on a fairly complex nonlinear simulation model. The outline of the paper is as follows: Section 2 provides a brief description of the dynamic model for a single-link flexible manipulator, and the effect of payload on the dynamic characteristics of the manipulator. Section 3 presents the nonlinear regulator problem. In section 4 the design of the nonlinear SDRE controller for a class of nonlinear control systems is explained. Section 5 sets up the problem of applying the nonlinear regulator to a single-link flexible manipulator. Control of flexible manipulator in the presence of varying payloads is investigated in Section 6. In section 7 simulation results for a single-link flexible manipulator are presented. Concluding remarks are given in section 8.

2. Dynamic model of flexible manipulator

The model of the flexible manipulator is obtained on basis of Lagrange's equations of motion [5], which may be written as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \quad i = 1, 2, 3 \tag{1}$$

where T is the kinetic energy, V potential energy, q_i generalized coordinate, and Q_i generalized force. The application of Lagrange's yields two sets of equations. The first set is associated with the rigid body degree of freedom defined by θ, and the other set is associated with the elastic degrees of freedom defined by δ_i. These two sets of equations of motion for a single-link flexible manipulator, are nonlinear time varying coupled, second order ordinary differential equations. The generalized coordinates are shown in Figure 1 for the single-link flexible manipulator. Under the Assumed Modes Method and retaining a finite number, $m = 2$ of modes, the dynamic equations for the flexible-link manipulator are derived as:

$$M(\theta,\delta_1,\delta_2)\begin{bmatrix}\ddot{\theta}\\ \ddot{\delta_1}\\ \ddot{\delta_2}\end{bmatrix}+\begin{bmatrix}h_1\left(\dot{\theta},\delta_1,\dot{\delta_1},\delta_2,\dot{\delta_2}\right)+F(\dot{\theta})\\ h_2(\dot{\theta},\delta_1,\delta_2,\dot{\delta_2})\\ h_3(\dot{\theta},\delta_1,\delta_2,\dot{\delta_1})\end{bmatrix}+\begin{bmatrix}0&0&0\\0&k_1&0\\0&0&k_2\end{bmatrix}\begin{bmatrix}\theta\\ \delta_1\\ \delta_2\end{bmatrix}$$
$$+\begin{bmatrix}D_1&0&0\\0&D_2&0\\0&0&D_3\end{bmatrix}\begin{bmatrix}\dot{\theta}\\ \dot{\delta_1}\\ \dot{\delta_2}\end{bmatrix}=\begin{bmatrix}u\\0\\0\end{bmatrix} \quad (2)$$

where $\delta = [\delta_1, \delta_2]^T \in \mathcal{R}^2$ is the deflection vector, $\theta \in \mathcal{R}$ is the joint variable, M represents the inertia matrix, $h = [h_1, h_2, h_3]^T$ represents the vector of the Coriolis and centrifugal forces, F is the Coulomb friction, u is the control input torque,

$$D = \begin{bmatrix}D_1&0&0\\0&D_2&0\\0&0&D_3\end{bmatrix} \in \mathcal{R}^{(m+1)\times(m+1)}$$

represents the viscous structural damping matrix, and

$$K = \begin{bmatrix}k_1&0&0\\0&k_2&0\\0&0&k_3\end{bmatrix} \in \mathcal{R}^{(m+1)\times(m+1)}$$

represents the stiffness matrix. Integer m is the number of flexible modes (or equivalently the number of mode shape functions), in our model $m = 2$. The main objective is to control the tip position of a single-link flexible manipulator whose dynamic specifications are derived in the Appendix. Assuming that the beam deflection d is small compared to the link length L, the normalized output may be written as $y = \theta + (d/l)$ with:

$$d = \sum_{i=1}^{m} \alpha_i \phi_i(l)\delta_i$$

where $\phi_i(l)$ represents the i^{th} mode shape and α_i represents a constant which when defining the normalized tip position (denoted by $y_t(t)$) is set to $\alpha_i = 1$. Therefore, $y_t(t)$ is given by:

$$y_t(t) = \theta + \frac{1}{l}\sum_{i=1}^{m} \phi_i(l)\delta_i \quad (3)$$

For the purpose of design, simulation, and control the dynamic equations of flexible-link manipulator (2) can be represented in the state-space model form. A state vector is first defined $x(t) = [x_1(t), x_2(t), \cdots, x_6(t)]^T$ where,

$$[x_1(t), x_2(t), \cdots, x_6(t)]^T = [\theta, \dot{\theta}, \delta_1, \dot{\delta_1}, \delta_2, \dot{\delta_2}]^T$$

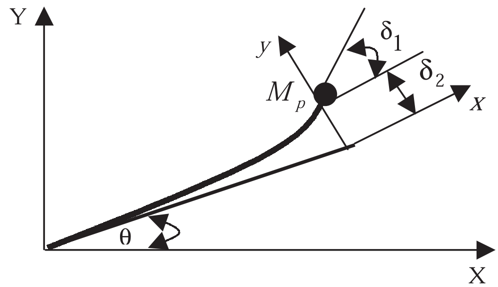

Figure 1. Geometric and Generalized Coordinates of a flexible link

Therefore, model (2) may be written as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_2 \\ L_1(x,u) \\ x_3 \\ L_2(x,u) \\ x_5 \\ L_3(x,u) \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} L_1(x,u) \\ L_2(x,u) \\ L_3(x,u) \end{bmatrix} = [M]^{-1} \begin{bmatrix} -h_1(x_2,x_3,x_4,x_5,x_6) - D_1x_2 - F(x_2) + u \\ -h_2(x_2,x_3,x_4,x_5,x_6) - D_2x_4 - K_1x_3 \\ -h_3(x_2,x_3,x_4,x_5,x_6) - D_3x_6 - K_2x_5 \end{bmatrix} \tag{5}$$

Detailed description of the coefficients in this equation is provided in the Appendix.

3. The nonlinear regulator problem

In the nonlinear regulator problem the aim is to minimize the infinite horizon cost function of the form:

$$J = \frac{1}{2}\int_{t_0}^{\infty} \left[x^TQ(x)x + u^TR(x)u\right] dt \tag{6}$$

with respect to the state x and control u subject to the nonlinear constraint

$$\dot{x} = a(x) + b(x)u. \tag{7}$$

Given state $x \in \mathcal{R}^n$ and control $u \in \mathcal{R}^m$, with $a, b, R, Q \in \mathcal{C}^k$, $k \geqslant 1$, where $Q(x) = H^T(x)H(x) \geqslant 0$, and $R(x) > 0$ for all x. It is assumed that $a(0) = 0$ so that the origin is an equilibrium point of the open loop system. We seek a stabilizing solution in the form $u = L(x)x$ where the nonlinear feedback gain L is a matrix function of x. The above formulation is analogous to linear quadratic regulator (LQR) theory [11] except that the matrices Q, R and L have all elements that are allowed to be functions of the state x. The SDRE method hinges on being able to write the constraint dynamics (7) in a point-wise linear structure, having state-dependent coefficient (SDC) form.

$$\dot{x} = A(x) + B(x)u \tag{8}$$

So that $a(x) = A(x)x$ and $b(x) = B(x)$, it is also known that there are an infinite number of ways to bring the nonlinear system to SDC form. Associated with the SDC form, we have the following definitions.

- $\{H(x), A(x)\}$ is an observable detectable parameterization of the nonlinear system (in a region Ω) if the pair $\{H(x), A(x)\}$ is point-wise observable (detectable) in the linear sense for all $x \in \Omega$.
- $\{A(x), B(x)\}$ is controllable stabilizable parameterization of the nonlinear system (in a region Ω) if the pair $\{A(x), B(x)\}$ is point-wise controllable (stabilizable) in the linear sense for all $x \in \Omega$.

4. SDRE nonlinear regulator

The SDRE approach for obtaining a suboptimal solution of the problem (6)-(7) is:

- Use direct parametrization to bring the nonlinear dynamics to the form of SDC (8).
- Solve the state-dependent Riccati equation (SDRE), to obtain $P(x)$:

$$A^T(x)P(x) + P(x)A(x) - P(x)B(x)R^{-1}(x)B^T(x)P(x) + Q(x) = 0 \tag{9}$$

  Accepting only $P(x) = P^T(x) \geqslant 0 \ \ \forall x$.

- Construct the nonlinear feedback controller gain via

$$u = -R^{-1}(x)B^T(x)P(x)x \tag{10}$$

These equations can be solved analytically to produce an equation for each element of u, or solved numerically at a sufficiently high sampling rate. It is quite clear from (10), that we must have full state feedback available in order to construct the control u using this method. The local stability of the closed loop system resulting from using the SDRE nonlinear regulator technique is determined by the following theorems from [3].

Theorem 1. *In addition to $a, b, R, Q \in \mathcal{C}^k, \quad k \geqslant 1$, assume that $A \in \mathcal{C}^k$, where A implies a globally stabilizable and detectable state dependent coefficient parametrization of the nonlinear system. Then the SDRE nonlinear regulation control method has a closed loop solution, which is locally asymptotically stable.*

This theorem is based on the fact that under these assumptions, the closed loop state-dependent solution is given by:

$$\dot{x} = \left[A(x) - B(x)R^{-1}(x)B^T(x)P(x)\right] \equiv F(x)x$$

where the closed loop function $F(x)$ is smooth and can be expanded in a Taylor series expansion about the origin to yield

$$\dot{x} = F(0)x + \Psi(x)||x|| \tag{11}$$

with $\lim_{||x||\to 0} \Psi(x) = 0$. In the neighborhood of the origin, the linear term, which has a constant, stable coefficient matrix, dominates the higher-order term yielding local asymptotic stability. Note that Theorem 1, relies on linearization arguments, and thus global stabilizability and detectability of the SDC parametrization are not required for local stability, but only stabilizability and detectability of the linearization of (7).

Theorem 2. *Consider (6) and (7) for scalar x, i.e., $n = 1$. Then there exists a unique SDC parameterization for $x \neq 0$, $A(x) = a(x)/x$, and the SDRE necessary condition for optimality $H_u = 0$, is always satisfied, where H is the Hamiltonian of the system given by*

$$H = \frac{1}{2}x^T Q(x)x + \frac{1}{2}u^T R(x)u + \lambda^T[A(x)x + B(x)u] \tag{12}$$

with stationary conditions:

$$\begin{aligned} &H_u^T = 0 \\ &\dot{\lambda} = -H_u^T \\ &\dot{x} = A(x)x + B(x)u. \end{aligned} \tag{13}$$

Theorem 2 assures us that the obtained solution is locally optimal whereas Theorem 1 will enable us to calculate the stability conditions for the flexible manipulator in sec. 6.

5. Application to a single-link flexible manipulator

To show the effectiveness of the proposed state-dependent Riccati equation technique, simulation studies on a flexible-link manipulator were performed. The effect of rotary inertia and shear deformation is ignored by assuming that the cross-sectional area of the link is small in comparison with its length. For the purpose of design; the dynamic equations of flexible-link manipulator (2) can be represented in the state-space model form. By choosing $[x_1, x_2, x_3, x_4, x_5, x_6]$ as the state vector and the tip position as the output, we can obtain the state-space equations (7) in the form:

$$\begin{aligned} &\dot{x}_1 = x_2 \\ &\dot{x}_2 = f_1(x_1, x_2, x_3, x_4, x_5, x_6) + g_1(x_3, x_5)u \end{aligned}$$

$$\begin{aligned}
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2(x_1, x_2, x_3, x_4, x_5, x_6) + g_2(x_3, x_5)u \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= f_3(x_1, x_2, x_3, x_4, x_5, x_6) + g_3(x_3, x_5)u \\
y &= x_1 + K_{tip}(C_1 x_3 + C_2 x_5)
\end{aligned} \tag{14}$$

where K_{tip}, C_1, C_2 are constants, depending on the arm characteristics. Based on the results developed in the preceding sections, the nonlinear model of the flexible manipulator has a general form

$$\dot{x} = f(x) + g(x)u \tag{15}$$

where $x = [x_1, x_2, x_3, x_4, x_5, x_6]^T$, $f(x)$ and $g(x)$ are vectors with nonlinear elements. Those vector functions can be calculated from equation (5) taking into account the formulae from the Appendix. The procedure for designing a nonlinear regulator for a flexible-link manipulator is as follows:

Step 1. Construct the state space model as in (15).

Step 2. Parameterize (15) in SDC form (8).

Step 3. Solve the SDRE algebraic Riccati equation (9) for P.

Step 3. Construct the nonlinear SDRE feedback controller (10).

6. Simulation and discussion

As mentioned earlier, the main objective is to control the tip position of a single link flexible manipulator y. Simulations were performed in Matlab/Simulink using Runge-Kutta, fourth-order numerical integration to implement and design the nonlinear controller. The purpose of the simulation is to demonstrate the performance of the developed model and controller algorithm in analyzing the effects of manipulator flexibility, and payload on the dynamic behavior of the system.

A simulated example is described in this section. To implement this, consider that the flexible link rotates on the horizontal plane i.e., the axis of rotation is vertical, the geometric and mass properties of the flexible manipulator are representing in Table 1. Before developing the control design, we study the open loop response of the flexible manipulator system. The flexible manipulator is excited with a bang-bang input torque profile of amplitude 1 $[Nm]$, shown in Figure 2. This is applied at the hub of the manipulator. The system variables considered here are: the joint angle θ, the tip deflection ν, and the tip position y as shown in Figure 2. The effect of the payload (fixed at the free end) has been investigated, by calculating the dynamic response The manipulator is

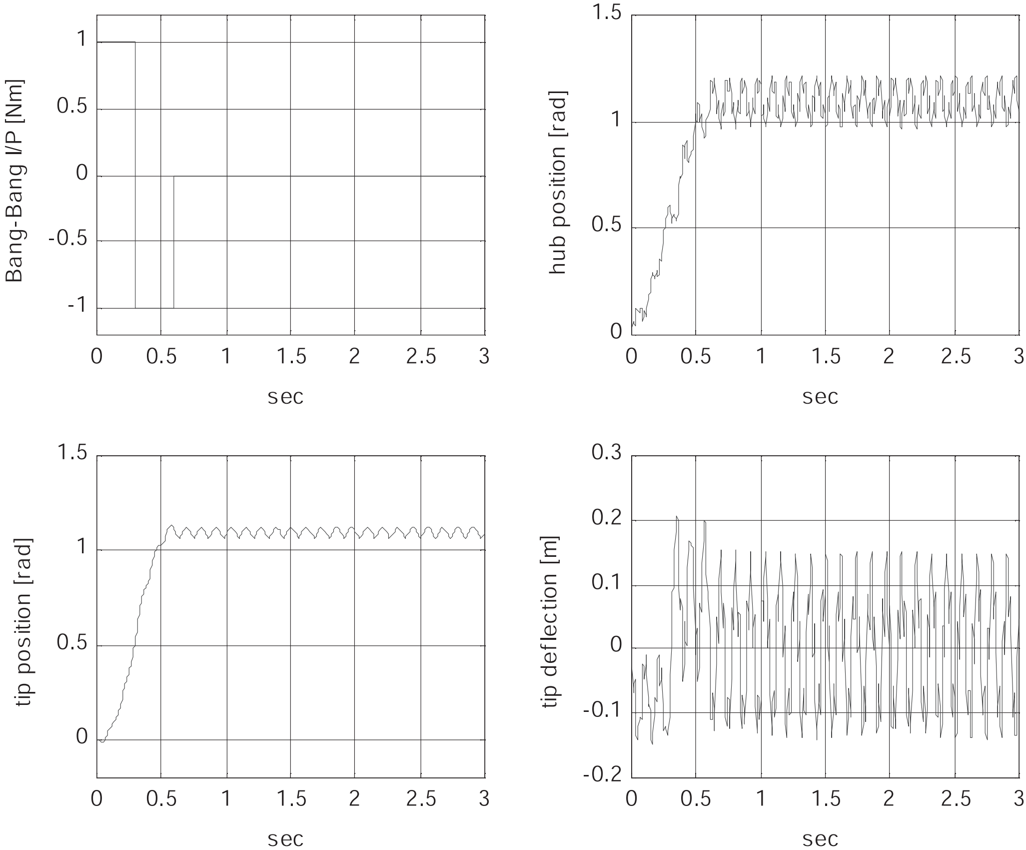

Figure 2. Bang-Bang input torque, hub and tip displacement, and tip deflection of the manipulator without payload

assumed to perform horizontal plane rotation with bang-bang torque profile (similar to that in Figure 2).

Table 1. Geometric and mass properties of the flexible link

Parameter	Value
Arm length l	1.0 $[m]$
Width	0.03 $[m]$
Thickness	0.003 $[m]$
Cross-Section area A	$9 \cdot 10^{-9}$ $[m^2]$
Young's Modulus E	$2 \cdot 10^{11}$ $[Nm]$
Specified mass ρ	7842 $[Kg/m^3]$
Area moment of inertia I_z	$20 \cdot 10^{-11}$ $[m^4]$

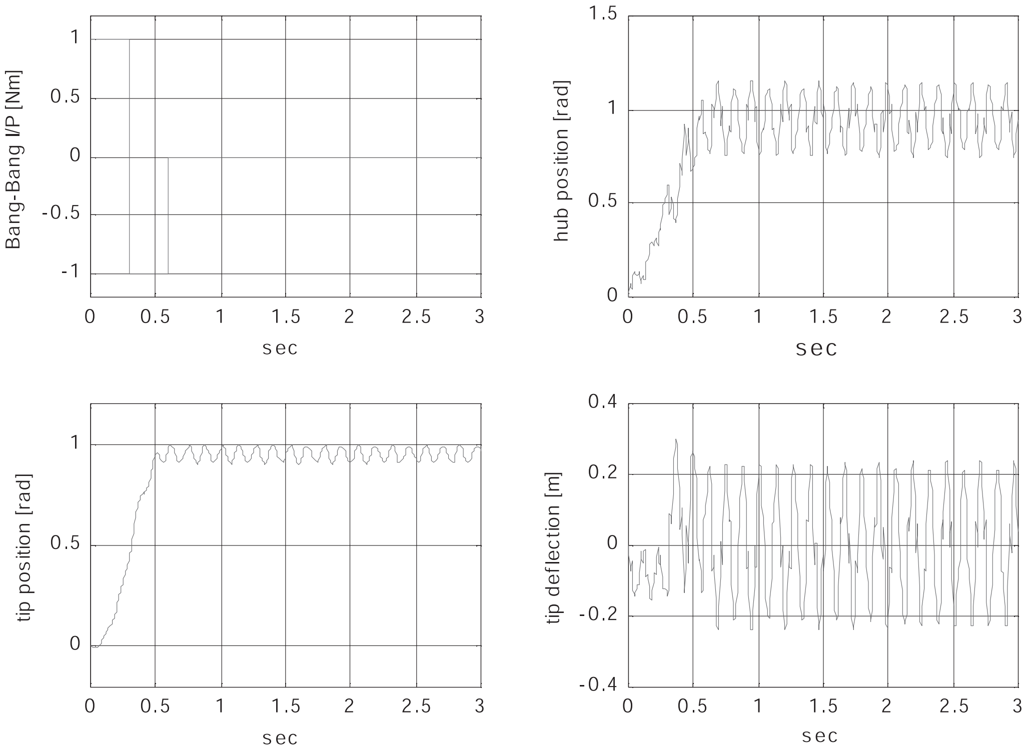

Figure 3. Bang-Bang input torque, hub, and tip displacement and tip deflection of the manipulator with payload $m_p/m = 0.05$

Figure 3, represents a sample of the dynamic response for $m_p/m = 0.05$. It can be seen that the increase of the payload will be accompanied by an increase of the elastic displacement and the residual vibration after performing a maneuver as shown in Figure 3. In the stability analysis, we consider the unforced system (i.e. $u = 0$):

$$\dot{x} = f(x). \tag{16}$$

We shall examine the stability of this system around the origin via the techniques of linearization and first integrals [11], by the so-called Principle of Stability in the First Approximation. If we assume that $f(x)$ is at least twice continuously differentiable and that the equilibrium of interest is the origin, then the following statements can be made about the local asymptotic stability of (15) based on the linear approximation of $f(x)$ at $x = 0$:

$$J_0 = \left[\frac{\partial f}{\partial x}\right]_{x=0} \tag{17}$$

Denote the Jacobian matrix of $f(x)$ at $x = 0$. Then if

- All the eigenvalues of J_0 have negative real parts, the origin is a locally asymptotically stable equilibrium of (15),

- At least one eigenvalue of J_0 has a positive real part; the origin is an unstable equilibrium of (15).

The Principle of Stability in the First Approximation obviously does not cover all cases of interest. In particular, it provides no information when all of the real parts of the eigenvalues of J_0 are nonpositive, and at least one eigenvalue has a zero real part. When this is the case, "Center Manifold Theory" may often be used to draw conclusions regarding the local stability properties of an equilibrium point for a time-invariant system. We also know that for any SDC dynamic parameterization $f(x) = A(x)x$ of (15), $A(0)$ must equal the Jacobian of f evaluated at zero, so that this necessary condition becomes that the pair $\{J_0, B(0)\}$ is stabilizabe.

Computing the Jacobian J_0 of (16) at the origin we find:

$$J_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100087 & 0 & -41984.7 & 0 & 9249.11 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -452323 & 0 & 188890 & 0 & -49233.3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of J_0 are:$\{0, 0, 0 \pm 60.45i, 0 \pm 295.9i\}$. We therefore attempt to verify the stability of (15) via linear feedback by the "Center Manifold Theory", under an arbitrary linear feedback, $u = -Kx$, where:

$$K = \begin{bmatrix} 44.52 & 3.6 & 33.58 & 8.61 & 0.453 & 0.1 \\ 3.6 & 1.13 & 1.35 & 2.7 & 0.031 & 0.053 \\ 33.58 & 1.35 & 1035.83 & 3.26 & 135.625 & 0.15 \\ 8.61 & 2.7 & 3.26 & 6.64 & -0.053 & 0.757 \\ 0.45 & 0.3 & 135.62 & -0.05 & 20212.89 & -0.499 \\ 0.1 & 0.05 & 0.15 & 0.75 & -0.49 & 2.841 \end{bmatrix}$$

The eigenvalues of the closed loop system are:

$$\left\{ \begin{array}{c} -3.605 \cdot 10^7 \\ -3.39 \\ -0.112 \pm 86.6i \\ -0.357 \pm 17.6i \end{array} \right\},$$

we find that (15) is asymptotically stabilizable by linear full state feedback $u = -Kx$. Thus, we have well motivated the need to seek nonlinear feedback to stabilize (15), which we now proceed to do. Based on the results obtained in the preceding sections, and the simulation results from the dynamic model of the single-link flexible manipulator, the nonlinear SDRE controller technique was designed and implemented in Matlab/Simulink to control the output of the flexible-link manipulator described above.

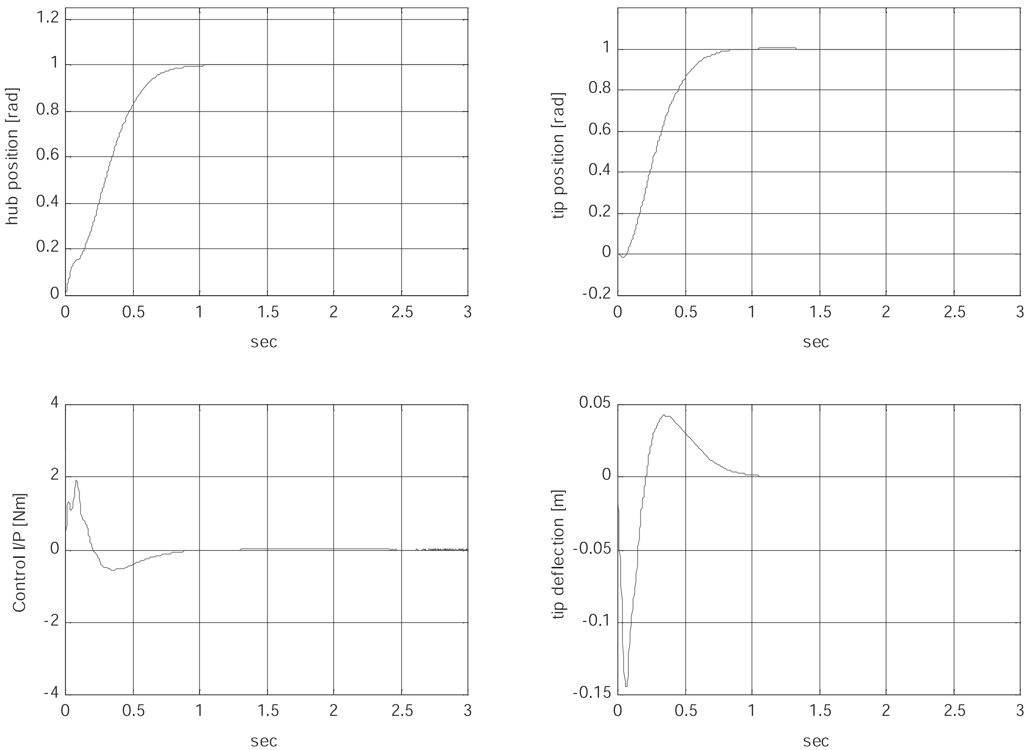

Figure 4. Step response of the flexible manipulator using SDRE controller without payload

We report here some simulation results obtained for the single-link flexible arm described above via SDRE controller technique. Figure 4 shows the closed loop output response of the tip position, and tip deflection for a step input with amplitude of 1.0 [rad]. For this simulation, the following weights R and Q were assumed:

$$R = 20, \qquad Q = \begin{bmatrix} 150 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 80G & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 80G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

where G is the first element in the inertia matrix M. As can be observed, a considerable good tracking, and smallest settling time of the tip position for the step input is achieved. The tip deflection is completely damped after 0.65 [sec]. For purpose of comparison Figure 5, shows the hub displacement, tip displacement, control input torque, and tip deflection of the flexible manipulator system for three cases of SDRE design with different state, and control weightings. Indeed, we see that the SDRE nonlinear regulator is an effective way of direct handling of unstable non-minimum phase systems, the simple way to adjust the control and state weighting matrices, and also offers significant design flexibility while yielding closed loop stability.

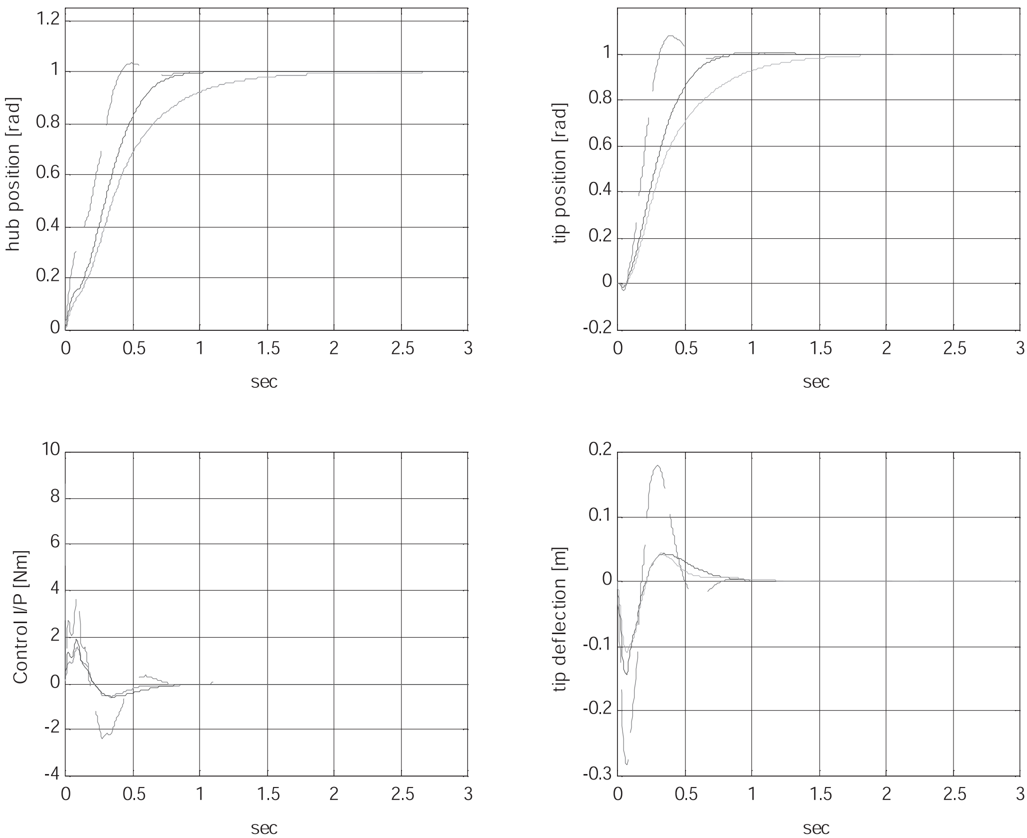

Figure 5. Step response of the flexible manipulator using SDRE controller with varying weighting

7. Control of flexible manipulator in the presence of varying payloads

In previous sections control of the flexible manipulator was set without the effect of payload at the free end. However, the payload is a very important parameter for the design and control of a flexible manipulator. Changes in payload mass result in changes in the dynamic performance of the arm, an important objective of the manipulator mechanical and control design is to increase its payload. The effect of the payload has been investigated for open loop torque control profile in Section 6, by calculating the dynamic response of the manipulator assuming different payload to manipulator mass ratio, m_p/m Figures (2,3). It is anticipated, however, that the increase of payload will be accompanied by an increase of the elastic displacement and the residual vibration after performing a maneuver. In this section we apply the nonlinear SDRE controller to control the flexible manipulator system for three-different ratios, of the payload mass to the mass of the arm $m_p/m = 0, 0.25, 0.5$, where m_p – mass of the payload, m – mass of the flexible link.

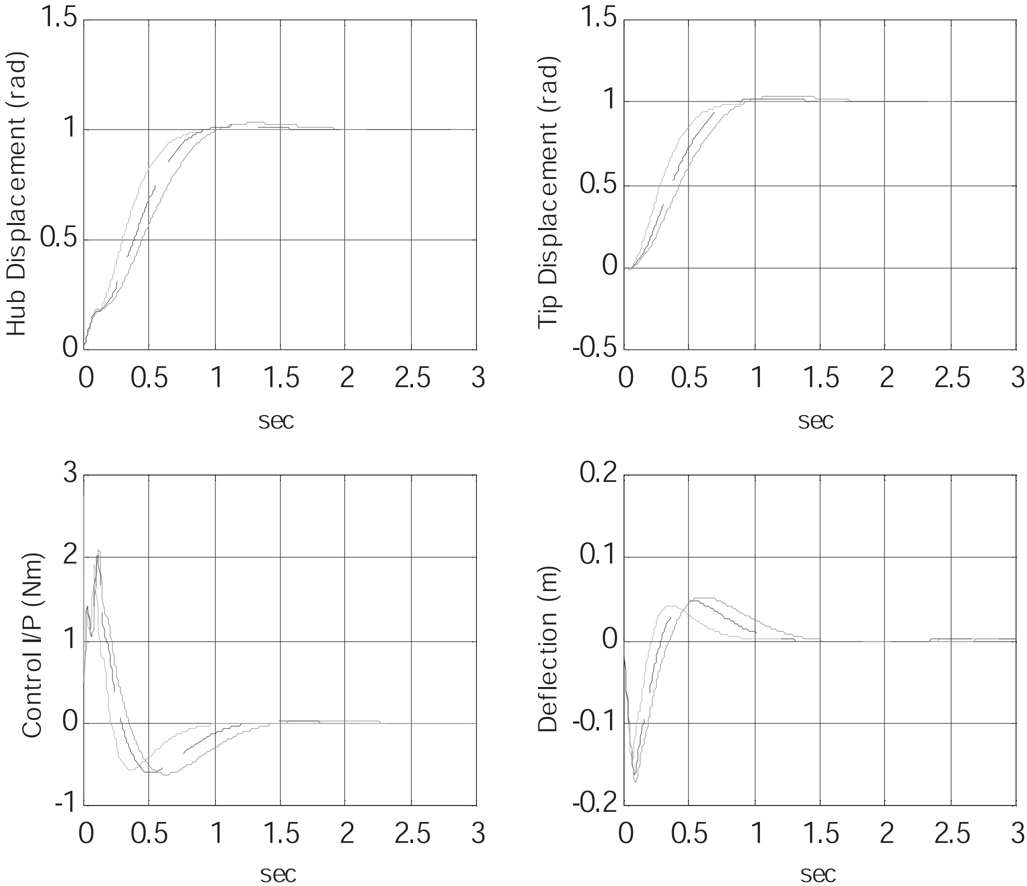

Figure 6. Step response of the flexible manipulator using SDRE controller with $m_p/m = 0, 0.25, 0.5$ payload

For the purpose of comparison, we use the same control and state weighting R and Q equation (18) respectively: as for case without payload in previous section. But we now change the fixed payload mass at the free end of the flexible manipulator. To facilitate comparison between cases, we start the simulations from the same initial conditions, we again sample at 0.005 [sec], and we use $A(x)$ parameterization given by (5) for all simulation cases. As a final basis of comparison, we show the plots for the three cases in Figure 6. In this figure, the solid lines represents the case without payload, dashed lines represent the case $m_p/m = 0.25$, and doted lines represent case $m_p/m = 0.5$. As expected, all of the outputs of the hub displacement, and tip displacement of the three cases are asymptotically approaching the value of one as desired with slightly more oscillations at the tip end of the flexible manipulator.

Two things are immediately apparent from the figure. Note first of all that increasing the mass ratio, m_p/m, increases the settling time. The second thing is the increase in the amplitude of oscillations as the ratio m_p/m increase. However, overall, the increase in the payload was handled sufficiently well by the SDRE controller.

8. Conclusion

The Lagrange mechanics and the assumed mode method have been used to derive a proposed dynamic model of a single link flexible manipulator having a revolute joint. The model may be used to investigate the motion of the manipulator in the horizontal, and vertical planes. The proposed model has been used to investigate the effect of two main design parameters, the payload, and the open loop control torque profile. The results of our investigation show that as long as the rest-to-rest rotational maneuver is considered, the payload has a dominant effect on the elastic deflection of the manipulator.

In this paper we focused on providing a theoretical basis for control of nonlinear systems via the state feedback SDRE techniques, which, have proven quite successful in a number of simulated applications, including the control of single link flexible manipulator. Extra design degrees of freedom arising from the non-uniqueness of the SDC parameterization can be utilized to enhance controller performance and the SDRE method does not cancel beneficial nonlinearities. It is shown that the proposed model and controller, under certain relatively mild conditions, renders the origin a globally asymptotically stable equilibrium point. Additional results in the paper show that the regulator is near optimal.

Throughout this paper, it was assumed that all the states of the plant are available for measurement. Obviously some of these states are available via standard sensors (such as hub angle, hub velocity and tip position). Other states may require more sophisticated sensors or observers.

Appendix

The elements of the dynamic equation (4) are as follows:

$$M=\begin{bmatrix} \rho A\begin{pmatrix}\left(\frac{l^3}{3}+\delta_1^2 A_3+2\delta_1\delta_2 A_4+\delta_2^2 A_5\right)\\ +m_p(4C_1^2\delta_1^2+8C_1C_2\delta_1\delta_2\\ +4C_2^2\delta_2^2+l^2)\end{pmatrix} & \begin{pmatrix}\rho A(A_1+lA_6)\\ +m_p(2C_1 l)\end{pmatrix} & \begin{pmatrix}\rho A(A_2+lA_7)\\ +m_p(2C_2 l)\end{pmatrix}\\ \begin{pmatrix}\rho A(A_1+lA_6)+\\ m_p(2C_1 l)\end{pmatrix} & \begin{pmatrix}\rho AA_3+\\ m_p(4C_1^2)\end{pmatrix} & \begin{pmatrix}\rho AA_4+\\ m_p(4C_1C_2)\end{pmatrix}\\ \begin{pmatrix}\rho A(A_2+lA_7)+\\ m_p(2C_2 l)\end{pmatrix} & \begin{pmatrix}\rho AA_4+\\ m_p(4C_1C_2)\end{pmatrix} & \begin{pmatrix}\rho AA_5+\\ m_p(4C_2^2)\end{pmatrix}\end{bmatrix}$$

$$\begin{bmatrix} h_1\\ h_2\\ h_3\end{bmatrix}=\begin{bmatrix}\begin{pmatrix}\rho A(2A_3\delta_1+2A_4\delta_2)+m_p(2C_1^2\delta_1+2C_1C_2\delta_1)+\\ \rho A(2A_4\delta_1+2A_5\delta_2)+m_p(8C_2^2\delta_2+8C_1C_2\delta_1)\end{pmatrix}\\ \left(-\rho AA_3-m_p(4C_1^2)\right)+\left(-\rho AA_4-m_p(4C_1C_2)\right)\\ \left(-\rho AA_4-m_p(4C_1C_2)\right)+\left(-\rho AA_5-m_p(4C_1^2)\right)\end{bmatrix}$$

$$D=\begin{bmatrix}0 & 0 & 0\\ 0 & 2\xi_1\omega_1\left[\rho A(A_3)+m_p(4C_1^2)\right] & 0\\ 0 & 0 & 2\xi_2\omega_2\left[\rho A(A_5)+m_p(4C_2^2)\right]\end{bmatrix}$$

The natural modal functions corresponding to clamped-free uniform beam are used here. These modal functions are given by:

$$Y_i(x)=C_i(\cos(\beta_i x)+\cosh(\beta_i x))+\sin(\beta_i x)+\sinh(\beta_i x)\qquad i=1,2$$

where

$$C_i=\frac{\sin(\beta_i l)+\sinh(\beta_i l)}{\cos(\beta_i l)+\cosh(\beta_i l)}\qquad i=1,2$$

$\beta_i l$ is the solution of the characteristic equation,

$$\beta_1 l=1.875104069\qquad \beta_2 l=4.694091133$$

$$\beta^4=\frac{\omega^2\rho A}{EI_z}$$

where ω is the natural frequency, and ξ is the modal damping ratio for the i_{th} normal mode.

$$A_1=\int_{-l}^{0} xY_1dx,\quad A_2=\int_{-l}^{0} xY_2dx,\quad A_3=\int_{-l}^{0} xY_1^2dx,\quad A_4=\int_{-l}^{0} Y_1Y_2dx,$$

$$A_5 = \int_{-l}^{0} Y_1^2 dx, \quad A_6 = \int_{-l}^{0} Y_1 dx, \quad A_7 = \int_{-l}^{0} Y_2 dx, \quad A_8 = \int_{-l}^{0} \left(\frac{\partial^2 Y_1}{\partial x^2}\right) dx,$$

$$A_9 = \int_{-l}^{0} \left(\frac{\partial^2 Y_1}{\partial x^2}\right)\left(\frac{\partial^2 Y_1}{\partial x^2}\right) dx, \quad A_{10} = \int_{-l}^{0} \left(\frac{\partial^2 Y_2}{\partial x^2}\right) dx$$

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