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From Continuous to Discrete Models of Linear Repetitive Processes

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Differential linear repetitive processes are a distinct class of 2D linear systems which pose problems which cannot (except in a few very restrictive special cases) be solved by application of existing linear systems theory, and hence by the use of many of the currently available tools for computer aided analysis and simulation. One such problem area is the construction of accurate numerically well conditioned discrete approximations of the dynamics of differential processes which could, as one example of a number of immediate applications areas, form the basis for the digital implementation of control laws. In this paper, we undertake a detailed investigation of the critical problems which arise when attempting to construct useful (for onward asnalysis/design studies) discrete approximations of the dynamics of differential linear repetitive processes and develop solutions to them. Numerical examples to support the results obtained are also given using a specially developed MATLAB based toolbox.

Key words: linear repetitive processes, discretization, MATLAB Toolbox, examples

1. Introduction

The essential unique characteristic of a repetitive process (also termed a multipass process in the early literature) can be illustrated by considering machining operations where the material or workpiece involved is processed by a sequence of passes of the processing tool. Assuming that the pass length α (i.e. the duration of a pass of the processing tool) is finite and constant, the output, or pass profile, $y_k(t),\ 0 \leqslant t \leqslant \alpha$, ($t$ being the independent spatial or temporal variable) produced on pass k acts as a forcing function on the next pass and hence contributes to the dynamics of the new pass profile $y_{k+1}(t),\ 0 \leqslant t \leqslant \alpha,\ k \geqslant 0$.

Industrial examples of repetitive processes include long–wall coal cutting and metal rolling operations [5], [31]. Also cases exist where adopting a repetitive process setting

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for analysis has major advantages over alternatives – so–called algorithmic examples. This is especially true for classes of iterative learning control schemes [1] and iterative solution algorithms for nonlinear dynamic optimal control problems based on the maximum principle [28]. In actual fact, iterative learning control as a subject area has its origins in the control of robotic systems where the problem for the control algorithm is to construct (or learn) the input that will generate the required output from the system under consideration. The mechanism of learning is that of repeated trials and the updating of the control inputs from trial to trial on the based of observed performance.

Publications dealing with robotics in an iterative learning control setting include [2], [4], [33] and the relevant cited references and the recent workshops [3], [18]. Learning optimal trajectories for non–holonomic systems, see, for example, [19], also has direct robotics implications. In [21], [26] it is shown that the stability theory for differential and discrete linear repetitive processes can be directly applied to iterative learning control schemes.

Repetitive processes clearly have a two–dimensional, or 2D, structure, i.e. information propagation occurs along a given pass (t direction) and from pass to pass (k direction). They are distinct from, in particular, the extensively studied 2D linear systems described by the Roesser [25] and Fornasini–Marchesini [6] state space models and the so–called 2D continuous-discrete linear systems (see, for example, [15]) by the fact that information propagation along the pass only occurs over a finite and fixed interval – the pass length α.

The basic unique control problem for repetitive processes is that the output sequence of pass profiles can contain oscillations that increase in amplitude in the pass to pass direction (i.e. in the k–direction in the notation for variables used here). Early approaches to stability analysis and controller design for (linear single–input single–output (SISO)) repetitive processes and, in particular, long–wall coal cutting [5] was based on first converting the underlying dynamics into those of an infinite–length single–pass process. This resulted, for example, in a scalar algebraic/delay system to which standard scalar inverse-Nyquist stability criteria could then be applied.

In general, however, it was soon established that this approach to stability analysis and controller design would, except in a few very restrictive special cases, lead to incorrect conclusions [20]. The basic reason for this is that such an approach effectively neglects their finite pass length repeatable nature and the effects of resetting the initial conditions before the start of each pass. To remove these difficulties, a rigorous stability theory has been developed [20], [24] based on an abstract model in a Banach space setting which includes all processes with linear dynamics and a constant pass length as special cases.

The development of this stability theory has led to a major programme of research aimed at producing a mature control systems theory for these processes for onward translation (where appropriate) into numerically reliable and implementable control algorithms/schemes. For a comprehensive treatment of the progress to date in this task see [26]. One major area which arises here is that of constructing discrete approximations to the dynamics of differential linear repetitive processes which clearly must have the required level of accuracy and be useful for onward analysis/design studies. In which context, it is already known from other work (see, for example, [12]) that (as expected intuitively) well known results/tools from the same area for standard, termed 1D here, linear systems cannot be applied. To address this basic difficulty, this paper details a range of discretization algorithms for differential linear repetitive processes (DLRP) which meet the criteria of accuracy and usefulness in terms of onward use. To support these algorithms, they have been included within a MATLAB based toolbox for linear repetitive processes (which is continually being updated) and here we use this software to give an illustrative example.

2. Problem formulation

The state space model of a differential linear repetitive process has the following form over $0 \leqslant t \leqslant \alpha,\ k = 0, 1, \ldots$ [28]

$$\begin{aligned} \dot{x}_{k+1}(t) &= \widehat{A}x_{k+1}(t) + \widehat{B}u_{k+1}(t) + \widehat{B}_0 y_k(t) \\ y_{k+1}(t) &= \widehat{C}x_{k+1}(t) + \widehat{D}u_{k+1}(t) + \widehat{D}_0 y_k(t). \end{aligned} \tag{1}$$

Here on pass k, $x_k(t)$ is the $n \times 1$ state vector, $y_k(t)$ is the $m \times 1$ vector pass profile, $u_k(t)$ is the $l \times 1$ vector of control inputs. To complete the process description, it is necessary to specify the initial or boundary conditions, i.e. the state initial vector on each pass and the initial pass profile. Here we assume the simplest possible form, i.e.

$$\begin{aligned} x_{k+1}(0) &= d_{k+1},\ k = 0, 1, \ldots \\ y_0(t) &= f(t),\ 0 \leqslant t \leqslant \alpha, \end{aligned} \tag{2}$$

where d_{k+1} is an $n \times 1$ vector with constant known entries and $f(t)$ is an $m \times 1$ vector whose entries are known functions of t.

Note. The process state space model (1) has the so–called unit memory property, i.e. it is only the pass profile on the previous pass which (explicitly) contributes to the current one. Non–unit memory linear repetitive processes are the natural generalization of (1) where a finite number, say $M > 1$, of previous pass profiles (explicitly) contribute to the current one. Such processes are not considered in this paper since the results given for the unit memory special case generalize in a natural manner.

The basic problem considered here is as follows:
Given a repetitive process of the form (1) and (2), construct a discrete approximation of the following form over $0 \leqslant p \leqslant \alpha,\ k = 0, 1, \ldots$

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p), \end{aligned} \tag{3}$$

with boundary conditions

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, k = 0, 1, \ldots \\ y_0(p) &= f(p), 0 \leqslant p \leqslant \alpha. \end{aligned} \tag{4}$$

The matrices in the discrete linear repetitive process state space model (3) are to be computed from those of (1) by formulas determined by the particular numerical approximation method used. The approximate solution generated by (3) and (4), should be as close as possible (in a well defined sense) to the exact solution obtained from (1) and (2) (assuming that it is known or may be calculated with negligible errors). Moreover, crucial system properties of (1) and (2), such as stability, should be preserved in (3) and (4) or conditions (which can be verified numerically) under which this is true should be available. (The reason for including explicit reference to the boundary conditions in both the differential and discrete cases here is that it is known, see [22] and [26], that the structure of the boundary conditions alone can destroy the stability properties of these processes. For example, if a process described by (1) and (2) is stable, this property may not hold if the pass state initial vector part of (2) is replaced by one which is an explicit function of points along the previous pass profile [22]).

3. Discretization Algorithms

Differential and discrete linear repetitive processes have clear structural links with 1D differential and discrete linear systems respectively. Hence a natural starting point for the development of discrete approximations to the dynamics of processes described by (1) is to consider the (direct) use of well known 1D methods. Hence we begin our analysis by considering, in turn, the Zero–Order–Hold (ZOH), forward, backward, and trapezoidal rules respectively. Where possible, we will propose modifications to these rules which improve their performance when applied in the repetitive process setting. This is followed by a similar analysis for higher order methods. Finally, all methods are compared both stand–alone and against each other on suitably chosen numerical examples.

3.1. ZOH approximation

The ZOH method is a very well known and commonly used method in 1D systems and gives very good/acceptable results in many cases. To apply this method to differential linear repetitive processes described by (1), however, it is necessary to assume that both the input and pass profile vectors are stepwise, i.e.

$$u_k(t+\tau) = u_k(t) \quad \text{for} \quad \tau \in [0, T) \tag{5}$$

$$y_k(t+\tau) = y_k(t) \quad \text{for} \quad \tau \in [0, T) \tag{6}$$

where T denotes the sampling period. Note also that the assumption of (6) has no 1D linear systems counterpart (in the current context, this is the first place we encounter the fact that linear repetitive processes cannot (in general) be studied by simply directly applying existing 1D linear systems theory or analysis tools).

Given (5) and (6), application of the ZOH method to (1) yields the following discrete time approximation for the current pass state vector

$$\begin{aligned} x_{k+1}(p+1) &= e^{\widehat{A}T}x_{k+1}(p) + \left\{\int_0^T e^{\widehat{A}\tau}\widehat{B}d\tau\right\}u_{k+1}(p) + \left\{\int_0^T e^{\widehat{A}\tau}\widehat{B}_0 d\tau\right\}y_k(p) \\ &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p), \end{aligned} \tag{7}$$

where (assuming $\widehat{A}$ is invertible)

$$\begin{aligned} A &= e^{\widehat{A}T} \\ B &= \widehat{A}^{-1}(e^{\widehat{A}T} - I)\widehat{B} \\ B_0 &= \widehat{A}^{-1}(e^{\widehat{A}T} - I)\widehat{B}_0. \end{aligned} \tag{8}$$

The pass profile updating equation in (1) remains unchanged under this approximation (its updating structure is discrete in the pass–to–pass direction).

Despite its appealing form and ease of numerical construction (a well defined algorithm whose numerical properties are well documented), the stepwise assumption on the entries in the pass profile vector may be too restrictive in many cases of practical interest [8], [12]. This is due to the fact that pass profile vector (y_k) represents the output of the repetitive process model and as such can not be fully controlled. Hence, a more realistic assumption to invoke is that only the entries in the input vector are stepwise, i.e. only (5) holds. In which case (7) is not valid and, as the numerical results later in this paper will demonstrate, the ZOH approximation without (6) gives very poor results.

3.2. Improvements to the ZOH approximation

For calculating integrals in (7) we must assume that the entries in both input and pass profile vectors are stepwise, i.e. (5) and (6) hold. As discussed in Section 3.1, this severely restricts the use of this method for the subject problem of this paper. As an alternative which retains the simplicity of the ZOH method, we can replace stepwise approximation with trapezoidal approximation. In the 2D linear systems domain, this approach was first proposed by [16], and Figure 1 summarizes the essential idea here. Instead of (6), we now write

$$y_k(t+\tau) = y_k(t) + [y_k(t+T) - y_k(t)]\frac{\tau}{T} \quad \text{for} \quad \tau \in [0, T), \tag{9}$$

and it is easy to see that the solution of the first equation in (1) can be written in the following general form

$$x_{k+1}(t+T) = e^{\widehat{A}T}x_{k+1}(t) + \int_0^T e^{\widehat{A}\tau}\widehat{B}u_{k+1}(t+\tau)d\tau + \int_0^T e^{\widehat{A}\tau}\widehat{B}_0 y_k(t+\tau)d\tau. \tag{10}$$

Then, after substituting (9) into (10) we obtain

$$\begin{aligned} x_{k+1}(t+T) &= e^{\widehat{A}T}x_{k+1}(t) + \int_0^T e^{\widehat{A}\tau}\widehat{B}d\tau u_{k+1}(t) + \int_0^T e^{\widehat{A}\tau}\widehat{B}_0 d\tau y_k(t) \\ &+ \int_0^T e^{\widehat{A}\tau}\widehat{B}_0\tau d\tau \frac{y_k(t+T)-y_k(t)}{T}, \end{aligned} \tag{11}$$

and now solving (11) (see also [16]), we obtain the discrete state space model

$$\begin{aligned} x_{k+1}(p+1) &= \bar{A}x_{k+1}(p) + \bar{B}u_{k+1}(p) + \bar{B}_0 y_k(p) + \bar{B}_1 y_k(p+1) \\ y_{k+1}(p) &= \bar{C}x_{k+1}(p) + \bar{D}u_{k+1}(p) + \bar{D}_0 y_k(p), \end{aligned} \tag{12}$$

where

$$\begin{aligned} \bar{A} &= e^{\widehat{A}T} \\ \bar{B} &= \widehat{A}^{-1}\left[e^{\widehat{A}T} - I\right]\widehat{B} \\ \bar{B}_0 &= \widehat{A}^{-1}\left[\frac{1}{T}\widehat{A}^{-1}(e^{\widehat{A}T} - I) - I\right]\widehat{B}_0 \\ \bar{B}_1 &= \widehat{A}^{-1}\left[e^{\widehat{A}T} - \frac{1}{T}\widehat{A}^{-1}(e^{\widehat{A}T} - I)\right]\widehat{B}_0 \\ \bar{C} &= \widehat{C}, \quad \bar{D} = \widehat{D}, \quad \bar{D}_0 = \widehat{D}_0. \end{aligned} \tag{13}$$

To remove the extra shifted term $y_k(p+1)$ in first equation of (12), define the state vector w_{k+1} as

$$w_{k+1}(p+1) = x_{k+1}(p+1) - \bar{B}_1 y_k(p+1). \tag{14}$$

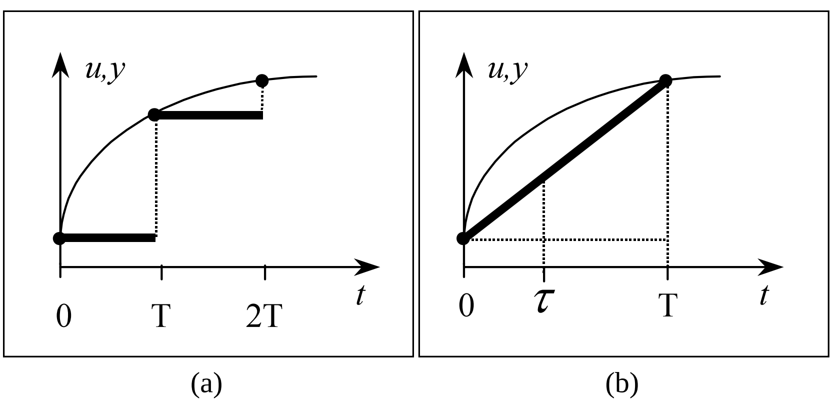


Figure 1. (a) stepwise approximation, (b) trapezoidal approximation.

Then, on writing (12) in terms of w, we obtain the following discrete linear repetitive process state space model

$$\begin{aligned} w_{k+1}(p+1) &= Aw_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) \\ y_{k+1}(p) &= Cw_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p), \end{aligned} \tag{15}$$

where

$$A = \bar{A}, \quad B = \bar{B}, \quad B_0 = \bar{B}_0 + \bar{A}\bar{B}_1, \quad C = \widehat{C}, \quad D = \widehat{D}, \quad D_0 = \widehat{D}_0 + \widehat{C}\bar{B}_1. \tag{16}$$

Note also that the introduction of (14) means that the state initial vector sequence $\{x_k(0)\}_{k\geqslant 1}$ in the target discrete linear repetitive process state space model must be replaced by

$$w_{k+1}(0) = x_{k+1}(0) - \bar{B}_1 y_k(0). \tag{17}$$

The stepwise assumption for input signal u (i.e. (5)) is exact in certain cases, e.g. when the input signal is a step function, but it can also be approximated by the trapezoidal rule, i.e.

$$u_k(t+\tau) = u_k(t) + [u_k(t+T) - u_k(t)]\frac{\tau}{T} \quad \text{for} \quad \tau \in [0, T), \tag{18}$$

and (10) should be evaluated in a similar manner to that detailed above.

In 1D linear systems theory, a basic property of the ZOH approximation is that if the original differential system is stable, i.e. all poles lie in the open left–half of the complex plane, then the resulting discrete linear system is also stable, i.e. all poles lie inside the unit circle in the complex plane. Next we ask the same question for the ZOH applied to linear repetitive processes. For this we need some basic results from the stability theory of linear constant pass length repetitive processes.

The stability theory [20], [28] for these processes consists of two distinct concepts, termed asymptotic stability and stability along the pass respectively. In effect, asymptotic stability is a form of bounded–input bounded–output (BIBO) stability in the sense that bounded input sequences (consisting of initial conditions, disturbances and control inputs which enter on each pass) produce bounded sequences of pass profiles over the, finite and constant by definition, pass length, where the bounded property is defined in terms of the norm on the underlying function space. Also asymptotic stability guarantees convergence in the pass–to–pass direction to a steady or so–called limit profile which, in effect, provides information on "transient" behaviour in the pass–to–pass direction.

In the case of the differential and discrete linear repetitive processes considered here, the necessary and sufficient condition for asymptotic stability is that $r(\widehat{D}_0) < 1$ and $r(D_0) < 1$ respectively. Also if this property holds, the resulting limit profile is described by a 1D differential (respectively discrete) linear systems state space model.

Hence if the example under consideration is asymptotically stable its repetitive dynamics can, after a "large number" of passes, be replaced by those of a 1D linear system (with obvious implications in terms of the structure and design of control schemes for these processes).

The fact that asymptotic stability property here is largely independent of the process dynamics and, in particular, of the eigenvalues of the matrices $\widehat{A}$ and A respectively, is due entirely to the finite pass length (over a finite duration even an unstable 1D linear system can only produce a bounded output), and means that it is possible for an asymptotically stable process to converge to a limit profile which is an unstable 1D linear system (for examples in both cases see [28]). This situation can be prevented by stability along the pass which demands the BIBO property uniformly, i.e. with respect to the pass length α. Also asymptotic stability is a necessary condition for stability along the pass.

As in the 1D linear systems case with the ZOH approximation, we would ideally wish to have the result that if the original differential process is stable (asymptotic or stability along the pass) then so is the resulting discrete approximation. Failing that, conditions under which the resulting discrete approximation is stable should be available.

In the case of the model of (15), it is immediate that this model could be asymptotically unstable (and hence unstable along the pass) even if the differential process from which it was derived is asymptotically stable. This is due to the fact that in constructing the discrete approximation the matrix $\widehat{D}_0$ is mapped to $\widehat{D}_0 + \widehat{C}\bar{B}_1$. Hence we see that preserving stability properties in the construction of discrete approximations to the dynamics of differential linear repetitive processes is somewhat more involved (see also later in this paper) than in the corresponding 1D linear systems case. This fact will also arise in some of the other discretization methods considered in this paper.

3.3. Forward approximation

One of the simplest possible methods available for the general problem considered in this paper is the forward difference approximation defined in terms of its action on a signal $x(t)$ by

$$x(pT) = x[(p-1)T] + T\dot{x}[(p-1)T]. \tag{19}$$

Application of this formula to the state equation in (1) yields

$$x_{k+1}(p+1) = x_{k+1}(p) + T[\widehat{A}x_{k+1}(p) + \widehat{B}u_{k+1}(p) + \widehat{B}_0 y_k(p)], \tag{20}$$

or, on also noting that the pass profile updating equation in (1) remains unchanged under this approximation,

$$x_{k+1}(p+1) = [I + \widehat{A}T]x_{k+1}(p) + \widehat{B}Tu_{k+1}(p) + \widehat{B}_0 T y_k(p) \tag{21}$$

$$y_{k+1}(p) = \widehat{C}x_{k+1}(p) + \widehat{D}u_{k+1}(p) + \widehat{D}_0 y_k(p). \tag{22}$$

Hence, the discrete linear repetitive process approximation of the dynamics of processes described by (1) is given by a model of the form (3) with matrices defined by

$$A = I + \widehat{A}T, \quad B = \widehat{B}T, \quad B_0 = \widehat{B}_0 T,$$

$$C = \widehat{C}, \quad D = \widehat{D}, \quad D_0 = \widehat{D}_0. \tag{23}$$

This resulting discrete approximation model is very simple but has the disadvantage that very small discretization periods are required to obtain acceptable (in the sense of any of the usual measures of the discretization errors) results. It does preserve asymptotic stability (since $D_0 = \widehat{D}_0$) but not stability along the pass. To establish this last fact, note that a necessary condition for stability along the pass of (1) is that all eigenvalues of the matrix $\widehat{A}$ have strictly negative real parts and for (3) that $r(A) < 1$.

3.4. Backward approximation

The backward approximation method is defined in terms of its action on a signal $x(t)$ by

$$x(pT) = x[(p-1)T] + T\dot{x}(pT), \tag{24}$$

and applying this formula to the state equation in (1) we obtain

$$x_{k+1}(p+1) = x_{k+1}(p) + T[\widehat{A}x_{k+1}(p+1) + \widehat{B}u_{k+1}(p+1) + \widehat{B}_0 y_k(p+1)]. \tag{25}$$

A key feature here is that (25) has extra shifted terms both in the current pass input vector $(u_{k+1}(p+1))$ and the current pass profile vector $(y_k(p+1))$ (we will also see such terms appearing in some of the other discretization methods considered later in this paper). Also, unless these terms can be removed by suitable transformation(s), the stability theory cannot be applied. Next we consider when such transformations actually exist.

Consider now the case when (5) and (6) hold. Then we can write (25) in the form

$$x_{k+1}(p+1) = x_{k+1}(p) + T[\widehat{A}x_{k+1}(p+1) + \widehat{B}u_{k+1}(p) + \widehat{B}_0, y_k(p)], \tag{26}$$

and this together with the unaltered equation for the pass profile vector updating leads to the model of (3) with the matrices A, B, B_0 and C, D, D_0 given by

$$\begin{aligned} A &= [I - \widehat{A}T]^{-1} \\ B &= [I - \widehat{A}T]^{-1}\widehat{B}T \\ B_0 &= [I - \widehat{A}T]^{-1}\widehat{B}_0 T \\ C &= \widehat{C}, \quad D = \widehat{D}, \quad D_0 = \widehat{D}_0. \end{aligned} \tag{27}$$

Note here that since $D_0 = \widehat{D}_0$, the resulting discrete approximation is asymptotically stable provided the original differential process has this property. The fact that $A \neq \widehat{A}$ means that stability along the pass of (3) does not necessarily imply that the discrete approximation has this property. In fact, using results from [28], the discrete approximation here is stable along the pass if, and only if, $r(\widehat{D}_0) < 1$, $r(A) < 1$, and

$$\det(C(z_1, z_2)) \neq 0, \ \forall \, (z_1, z) \in \overline{U}^2, \tag{28}$$

where

$$C(z_1, z) = \begin{bmatrix} I_n - z_1 A & -z_1 B_0 \\ -zC & I_m - zD_0 \end{bmatrix}, \tag{29}$$

and

$$\overline{U}^2 = \{(z_1, z) : |z_1| \leqslant 1, \ |z| \leqslant 1\}.$$

In the case when (5) and (6) not hold, we have that

$$x_{k+1}(p+1) = x_{k+1}(p) + T[\widehat{A}x_{k+1}(p+1) + \widehat{B}u_{k+1}(p+1) + \widehat{B}_0 y_k(p+1)], \tag{30}$$

or

$$x_{k+1}(p) = x_{k+1}(p+1) - \widehat{A}Tx_{k+1}(p+1) - \widehat{B}Tu_{k+1}(p+1) - \widehat{B}_0 Ty_k(p+1). \tag{31}$$

Now define the new state vector w_{k+1} as

$$w_{k+1}(p+1) \,\widehat{=}\, x_{k+1}(p). \tag{32}$$

Then

$$[I - \widehat{A}T]x_{k+1}(p+1) = \widehat{B}Tu_{k+1}(p+1) + \widehat{B}_0 Ty_k(p+1) + w_{k+1}(p+1), \tag{33}$$

and hence

$$\begin{aligned} x_{k+1}(p+1) &= [I - \widehat{A}T]^{-1}w_{k+1}(p+1) + [I - \widehat{A}T]^{-1}\widehat{B}Tu_{k+1}(p+1) \\ &+ [I - \widehat{A}T]^{-1}\widehat{B}_0 Ty_k(p+1). \end{aligned} \tag{34}$$

Since all terms in (34) are expressed in terms of $(p + 1)$, we also have that

$$\begin{aligned} x_{k+1}(p) &= [I - \widehat{A}T]^{-1}w_{k+1}(p) + [I - \widehat{A}T]^{-1}\widehat{B}Tu_{k+1}(p) \\ &+ [I - \widehat{A}T]^{-1}\widehat{B}_0 Ty_k(p), \end{aligned} \tag{35}$$

and substituting (35) into (32) gives

$$\begin{aligned} w_{k+1}(p+1) &= [I - \widehat{A}T]^{-1}w_{k+1}(p) + [I - \widehat{A}T]^{-1}\widehat{B}Tu_{k+1}(p) \\ &+ [I - \widehat{A}T]^{-1}\widehat{B}_0 Ty_k(p), \end{aligned} \tag{36}$$

and the output equation of (1) becomes, on substituting (35),

$$\begin{aligned} y_{k+1}(p) &= \widehat{C}[I - \widehat{A}T]^{-1}w_{k+1}(p) + \{\widehat{C}[I - \widehat{A}T]^{-1}\widehat{B}T + \widehat{D}\}u_{k+1}(p) \\ &+ \{\widehat{C}[I - \widehat{A}T]^{-1}\widehat{B}_0 T + \widehat{D}_0\}y_k(p). \end{aligned} \tag{37}$$

Hence the matrices in the discrete approximation (3) in this case are given by

$$A = [I - \widehat{A}T]^{-1}$$

$$\begin{aligned} B &= [I - \widehat{A}T]^{-1}\widehat{B}T \\ B_0 &= [I - \widehat{A}T]^{-1}\widehat{B}_0T \\ C &= \widehat{C}[I - \widehat{A}T]^{-1} \\ D &= \widehat{C}[I - \widehat{A}T]^{-1}\widehat{B}T + \widehat{D} \\ D_0 &= \widehat{C}[I - \widehat{A}T]^{-1}\widehat{B}_0T + \widehat{D}_0. \end{aligned} \tag{38}$$

The structure of these matrices in this case immediately shows that asymptotic stability (and hence stability along the pass) of the original differential process does not guarantee that the resulting discrete approximation has this property.

3.5. Trapezoidal approximation

This approximation is defined in terms of its action on a signal $x(t)$ as follows

$$x(pT) = x[(p-1)T] + \frac{T}{2}[\dot{x}(pT) + \dot{x}(p-1)T]. \tag{39}$$

Under the assumptions of (5) and (6), the dynamics of differential repetitive process (1) can be approximated by a discrete linear repetitive process of the form (3) were the matrices A, B, B_0, C, D, D_0 are given by

$$\begin{aligned} A &= \left[I + \frac{\widehat{A}T}{2}\right]\left[I - \frac{\widehat{A}T}{2}\right]^{-1} \\ B &= \left[I - \frac{\widehat{A}T}{2}\right]^{-1}\widehat{B}T \\ B_0 &= \left[I - \frac{\widehat{A}T}{2}\right]^{-1}\widehat{B}_0T \\ C &= \widehat{C}, \quad D = \widehat{D}, \quad D_0 = \widehat{D}_0. \end{aligned} \tag{40}$$

Note here that $D_0 = \widehat{D}_0$ and hence the discrete approximation is asymptotically stable provided the original differential process has this property. Conditions for stability along the pass of the discrete approximation can be obtained by applying (28) to the final state space model.

As before, the stepwise assumption on the entries in the pass profile vector (equation (6)) may be too restrictive in many cases of practical interest [12]. Hence, the more realistic assumption is only to assume that the entries in the input vector are stepwise, i.e. only (5) holds. In this case, applying the trapezoidal rule to (1) yields the following discrete linear repetitive process state space model

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0'y_k(p+1) + B_0'y_k(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0y_k(p), \end{aligned} \tag{41}$$

where matrices of the model are given in (40) with the exception that B_0 is replaced by B_0' where

$$B_0' = \frac{1}{2}(B_0). \tag{42}$$

The novel feature of this model is that the state equation contains an extra term whose index is shifted in the along the pass direction ($y_k(p+1)$) and hence is closer to the Fornasini–Marchesini model [6] for 2D discrete linear systems recursive in the positive quadrant.

To investigate this situation in more detail, it is first necessary to introduce the equivalent 2D Roesser–type model [25] description of the dynamics of a discrete linear repetitive process, where here we do this by first "forward shifting" the pass profile vector using the following substitution (which is well known in 1D analysis)

$$y_{k-1}(p) := Y_k(p),\ k \geqslant 0,\ 0 \leqslant i < \alpha, \tag{43}$$

and then "retarding" the pass index by setting $l := k+1$. Also define the new state vector $X_l(p)$ as

$$X_l(p) = x_l(p) - B_0'Y_l(p). \tag{44}$$

Then it follows immediately that a discrete linear repetitive process state space model of the form (41) has been transformed to the following form

$$X_l(p+1) = AX_l(p) + Bu_l(p) + \widetilde{B_0}Y_l(p) \tag{45}$$

$$Y_{l+1}(p) = CX_l(p) + Du_l(p) + \widetilde{D_0}Y_l(p), \tag{46}$$

where

$$\widetilde{B_0} = AB_0' + B_0' \tag{47}$$

$$\widetilde{D_0} = D_0 + CB_0' \tag{48}$$

and the other matrices have not been detailed here as they are not relevant to the following discussion.

The model of (45)–(48) (despite notational differences with the 2D systems literature) has the structure of a Roesser–type model. In the context of this paper, however, the main conclusion arising from (47) and (48), is that use of the trapezoidal rule to approximate the dynamics of a differential linear repetitive process under the assumption that only the entries in the input vector are stepwise does not, in general, mean that the discrete approximation of a stable differential process (asymptotic or along the pass) will also have this property.

A more detailed treatment on this model, together with supporting numerical examples, can be found in [10], [11], [12].

3.6. Improved trapezoidal approximation

To avoid the disadvantages detailed above of the standard trapezoidal method applied to linear differential repetitive processes, the following modification is proposed [10] (the basic idea is similar to the one often used in 1D systems analysis).

First apply the trapezoidal rule described by (39) and, after rearranging the resulting equation such that all $(p+1)$ terms are on the left and all (p) terms on the right, we have

$$\begin{aligned} x_{k+1}(p+1) - \frac{\widehat{A}T}{2}x_{k+1}(p+1) - \frac{\widehat{B}_0T}{2}y_k(p+1) - \frac{\widehat{B}T}{2}u_{k+1}(p+1) \\ = x_{k+1}(p) + \frac{\widehat{A}T}{2}x_{k+1}(p) + \frac{\widehat{B}_0T}{2}y_k(p) + \frac{\widehat{B}T}{2}u_{k+1}(p), \end{aligned} \tag{49}$$

and define the new state vector w_{k+1} as

$$w_{k+1}(p+1) \,\widehat{=}\, x_{k+1}(p) + \frac{\widehat{A}T}{2}x_{k+1}(p) + \frac{\widehat{B}_0T}{2}y_k(p) + \frac{\widehat{B}T}{2}u_{k+1}(p), \tag{50}$$

or

$$\begin{aligned} x_{k+1}(p) &= \left[I - \frac{\widehat{A}T}{2}\right]^{-1} w_{k+1}(p) + \left[I - \frac{\widehat{A}T}{2}\right]^{-1} \frac{\widehat{B}_0T}{2}y_k(p) \\ &+ \left[I - \frac{\widehat{A}T}{2}\right]^{-1} \frac{\widehat{B}T}{2}u_{k+1}(p). \end{aligned} \tag{51}$$

Substituting (51) into (50) now gives

$$\begin{aligned} &w_{k+1}(p+1) = \\ &\left[I - \frac{\widehat{A}T}{2}\right]^{-1} w_{k+1}(p) + \left[I - \frac{\widehat{A}T}{2}\right]^{-1} \frac{\widehat{B}_0T}{2}y_k(p) + \left[I - \frac{\widehat{A}T}{2}\right]^{-1} \frac{\widehat{B}T}{2}u_{k+1}(p) \\ &+\frac{\widehat{A}T}{2}\left\{\left[I-\frac{\widehat{A}T}{2}\right]^{-1} w_{k+1}(p) + \left[I-\frac{\widehat{A}T}{2}\right]^{-1}\frac{\widehat{B}_0T}{2}y_k(p) + \left[I-\frac{\widehat{A}T}{2}\right]^{-1}\frac{\widehat{B}T}{2}u_{k+1}(p)\right\} \\ &+\frac{\widehat{B}_0T}{2}y_k(p) + \frac{\widehat{B}T}{2}u_{k+1}(p), \end{aligned} \tag{52}$$

or, after an obvious rearrangement,

$$w_{k+1}(p+1) = \left[I + \frac{\widehat{A}T}{2}\right]\left[I - \frac{\widehat{A}T}{2}\right]^{-1} w_{k+1}(p)$$

$$+ \left[I - \frac{\widehat{A}T}{2}\right]^{-1} \widehat{B}Tu_{k+1}(p) + \left[I - \frac{\widehat{A}T}{2}\right]^{-1} \widehat{B}_0 Ty_k(p). \quad (53)$$

The pass profile vector updating equation in this case is given by

$$\begin{aligned} y_{k+1}(p) &= \widehat{C}\left[I - \frac{\widehat{A}T}{2}\right]^{-1} w_{k+1}(p) + \left\{\widehat{D} + \widehat{C}\left[I - \frac{\widehat{A}T}{2}\right]^{-1} \frac{\widehat{B}T}{2}\right\} u_{k+1}(p) \\ &+ \left\{\widehat{D}_0 + \widehat{C}\left[I - \frac{\widehat{A}T}{2}\right]^{-1} \frac{\widehat{B}_0 T}{2}\right\} y_k(p), \end{aligned} \quad (54)$$

and, finally, the matrices defining the discrete linear repetitive process state space model approximation of the differential process in this case are given by

$$\begin{aligned} A &= \left[I + \frac{\widehat{A}T}{2}\right]\left[I - \frac{\widehat{A}T}{2}\right]^{-1} \\ B &= \left[I - \frac{\widehat{A}T}{2}\right]^{-1} \widehat{B}T \\ B_0 &= \left[I - \frac{\widehat{A}T}{2}\right]^{-1} \widehat{B}_0 T \\ C &= \widehat{C}\left[I - \frac{\widehat{A}T}{2}\right]^{-1} \\ D &= \widehat{D} + \widehat{C}\left[I - \frac{\widehat{A}T}{2}\right]^{-1} \frac{\widehat{B}T}{2} \\ D_0 &= \widehat{D}_0 + \widehat{C}\left[I - \frac{\widehat{A}T}{2}\right]^{-1} \frac{\widehat{B}_0 T}{2}. \end{aligned} \quad (55)$$

Note also that the introduction of (50) means that the state initial vector sequence $\{x_k(0)\}_{k\geqslant 1}$ in the target discrete linear repetitive process state space model must be replaced by

$$w_{k+1}(0) = x_{k+1}(0) - \frac{\widehat{A}T}{2} x_{k+1}(0) - \frac{\widehat{B}_0 T}{2} y_k(0) - \frac{\widehat{B}T}{2} u_{k+1}(0). \quad (56)$$

The model defined by (53) and (54) does not require the step–wiseness assumptions (5) and (6) and does not have any advanced terms, such as those in (41), which is a main advantage of this approach. Note also that $D_0 \neq \widehat{D}_0$, and hence is not guaranteed that stability properties (asymptotic and along the pass) which hold for the differential process will also be present in the resulting discrete approximation.

3.7. Higher order approximation

In order to improve discretization quality (in the sense of discretization errors), this section considers the use of a higher order, single step method of numerical integration [17], [29]. One particular option here is the following defined which is defined in terms of its action on a signal $x(t)$ by

$$x[(p+1)T] = x(pT) + \frac{T}{2}[\dot{x}(pT) + \dot{x}(p+1)T] + \frac{T^2}{12}[\ddot{x}(pT) - \ddot{x}(p+1)T]. \quad (57)$$

Application of this method to (1) yields

$$\begin{aligned} &x_{k+1}(p+1) = x_{k+1}(p) \qquad (58)\\ &+\frac{T}{2}\left\{\widehat{A}x_{k+1}(p+1)+\widehat{B}u_{k+1}(p+1)+\widehat{B}_0y_k(p+1)+\widehat{A}x_{k+1}(p)+\widehat{B}u_{k+1}(p)+\widehat{B}_0y_k(p)\right\}\\ &-\frac{T^2}{12}\left\{\widehat{A}\dot{x}_{k+1}(p+1)+\widehat{B}\dot{u}_{k+1}(p+1)+\widehat{B}_0\dot{y}_k(p+1)-\widehat{A}\dot{x}_{k+1}(p)-\widehat{B}\dot{u}_{k+1}(p)-\widehat{B}_0\dot{y}_k(p)\right\}. \end{aligned}$$

Assuming also that the derivatives of the entries in the state and pass profile vectors are step-wise functions, i.e. the input and pass profile vector entries are piece-wise linear functions, enables us to rewrite (58) in the form

$$\begin{aligned} &x_{k+1}(p+1) = x_{k+1}(p) \qquad (59)\\ &+\frac{T}{2}\left\{\widehat{A}x_{k+1}(p+1)+\widehat{B}u_{k+1}(p+1)+\widehat{B}_0y_k(p+1)+\widehat{A}x_{k+1}(p)+\widehat{B}u_{k+1}(p)+\widehat{B}_0y_k(p)\right\}\\ &+\frac{T^2}{12}\left\{\widehat{A}\dot{x}_{k+1}(p) - \widehat{A}\dot{x}_{k+1}(p+1)\right\} \end{aligned}$$

and on using (1) we obtain

$$\begin{aligned} &x_{k+1}(p+1) = x_{k+1}(p) \qquad (60)\\ &+\frac{T}{2}\left\{\widehat{A}x_{k+1}(p+1)+\widehat{B}u_{k+1}(p+1)+\widehat{B}_0y_k(p+1)+\widehat{A}x_{k+1}(p)+\widehat{B}u_{k+1}(p)+\widehat{B}_0y_k(p)\right\}\\ &+\frac{T^2}{12}\left\{\widehat{A}^2x_{k+1}(p) + \widehat{A}\widehat{B}_0y_k(p) + \widehat{A}\widehat{B}u_{k+1}(p) - \widehat{A}^2x_{k+1}(p+1) - \widehat{A}\widehat{B}_0y_k(p+1)\right.\\ &\left.- \widehat{A}\widehat{B}u_{k+1}(p+1)\right\}. \end{aligned}$$

Finally, from (60) we obtain the following discrete model

$$\begin{aligned} x_{k+1}(p+1) &= A_1x_{k+1}(p) + B_1u_{k+1}(p+1) + B_2u_{k+1}(p)\\ &+ B_{01}y_k(p+1) + B_{02}y_k(p) \qquad (61)\\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0y_k(p), \qquad (62) \end{aligned}$$

where

$$\begin{aligned} A_1 &= \left[I - \frac{\widehat{A}T}{2} + \frac{\widehat{A}^2T^2}{12}\right]^{-1}\left[I + \frac{\widehat{A}T}{2} + \frac{\widehat{A}^2T^2}{12}\right] \\ B_1 &= \left[I - \frac{\widehat{A}T}{2} + \frac{\widehat{A}^2T^2}{12}\right]^{-1}\left[\frac{\widehat{B}T}{2} - \frac{\widehat{A}\widehat{B}T^2}{12}\right] \\ B_2 &= \left[I - \frac{\widehat{A}T}{2} + \frac{\widehat{A}^2T^2}{12}\right]^{-1}\left[\frac{\widehat{B}T}{2} + \frac{\widehat{A}\widehat{B}T^2}{12}\right] \\ B_{01} &= \left[I - \frac{\widehat{A}T}{2} + \frac{\widehat{A}^2T^2}{12}\right]^{-1}\left[\frac{\widehat{B}_0T}{2} - \frac{\widehat{A}\widehat{B}_0T^2}{12}\right] \\ B_{02} &= \left[I - \frac{\widehat{A}T}{2} + \frac{\widehat{A}^2T^2}{12}\right]^{-1}\left[\frac{\widehat{B}_0T}{2} + \frac{\widehat{A}\widehat{B}_0T^2}{12}\right] \\ C &= \widehat{C}, \quad D = \widehat{D}, \quad D_0 = \widehat{D}_0. \end{aligned} \tag{63}$$

3.8. Improved higher order approximation

In the model (61) there two additional terms relative to (3), i.e. $u_{k+1}(p+1)$ and $y_k(p+1)$. To remove these, we can apply a similar approach to that used for the trapezoidal method of Section 3.6. In particular, first note that (60) can be written as

$$\begin{aligned} x_{k+1}(p+1) - \frac{\widehat{A}T}{2}x_{k+1}(p+1) - \frac{\widehat{B}_0T}{2}y_k(p+1) - \frac{\widehat{B}T}{2}u_{k+1}(p+1) - \frac{\widehat{A}^2T^2}{12}x_{k+1}(p+1) \\ -\frac{\widehat{A}\widehat{B}_0T^2}{12}y_k(p+1) - \frac{\widehat{A}\widehat{B}T^2}{12}u_{k+1}(p+1) = x_{k+1}(p) + \frac{\widehat{A}T}{2}x_{k+1}(p) + \frac{\widehat{B}_0T}{2}y_k(p) \\ +\frac{\widehat{B}T}{2}u_{k+1}(p) + \frac{\widehat{A}^2T^2}{12}x_k(p) + \frac{\widehat{A}\widehat{B}_0T^2}{12}y_k(p) + \frac{\widehat{A}\widehat{B}T^2}{12}u_{k+1}(p). \end{aligned} \tag{64}$$

Then introducing the new state vector w_{k+1} as

$$\begin{aligned} w_{k+1}(p+1) &\widehat{=} x_{k+1}(p) + \frac{\widehat{A}T}{2}x_{k+1}(p) + \frac{\widehat{B}_0T}{2}y_k(p) + \frac{\widehat{B}T}{2}u_{k+1}(p) \\ &+ \frac{\widehat{A}^2T^2}{12}x_k(p) + \frac{\widehat{A}\widehat{B}_0T^2}{12}y_k(p) + \frac{\widehat{A}\widehat{B}T^2}{12}u_{k+1}(p), \end{aligned} \tag{65}$$

the discrete approximation in this case has the form of (3) with

$$\begin{aligned} A &= QP \\ B &= QPR + \frac{\widehat{B}T}{2} + \frac{\widehat{A}\widehat{B}T^2}{12} \end{aligned}$$

$$\begin{aligned} B_0 &= QPS + \frac{\widehat{B}_0 T}{2} + \frac{\widehat{A}\widehat{B}_0 T^2}{12} \\ C &= \widehat{C}P \\ D &= \widehat{D} + \widehat{C}PR \\ D_0 &= \widehat{D}_0 + \widehat{C}PS, \end{aligned} \tag{66}$$

where

$$\begin{aligned} P &= \left[I - \frac{\widehat{A}T}{2} + \frac{\widehat{A}^2 T^2}{12} \right]^{-1} \\ Q &= \left[I + \frac{\widehat{A}T}{2} + \frac{\widehat{A}^2 T^2}{12} \right] \\ R &= \left[\frac{\widehat{B}T}{2} - \frac{\widehat{A}\widehat{B}T^2}{12} \right] \\ S &= \left[\frac{\widehat{B}_0 T}{2} - \frac{\widehat{A}\widehat{B}_0 T^2}{12} \right]. \end{aligned} \tag{67}$$

Note also that the introduction of (65) means that the state initial vector sequence $\{x_k(0)\}_{k\geqslant 1}$ in the target discrete linear repetitive process state space model must be replaced by

$$\begin{aligned} w_{k+1}(0) &= \left[I - \frac{\widehat{A}T}{2} + \frac{\widehat{A}^2 T^2}{12} \right] x_{k+1}(0) - \left[\frac{\widehat{B}T}{2} - \frac{\widehat{A}\widehat{B}T^2}{12} \right] u_{k+1}(0) \\ &- \left[\frac{\widehat{B}_0 T}{2} - \frac{\widehat{A}\widehat{B}_0 T^2}{12} \right] y_k(0). \end{aligned} \tag{68}$$

Here again $D_0 \neq \widehat{D}_0$, and hence stability properties (asymptotic and along the pass) of the original differential process are not guaranteed to be present in the resulting discrete approximation.

4. Numerical examples

In this section we give two numerical examples in order to compare discretization methods detailed in Section 3. In this context, note that it is possible to display information concerning the output from discrete linear repetitive processes under a number of headings, where here a key feature of interest here is clearly the approximation errors introduced by a use of a particular discretization method. Here we will present the simulation results under the following categories:

1. In absolute terms where we will use 3–dimensional plots.
2. As a function of the sampling period (T) or as a function of a particular discretization method used. In this case we will use 2–dimensional plots.

In the 3–dimensional plots we graph absolute errors, i.e. the curves generated by

$$y_k^e(p) := |\widehat{y}_k(p) - \widetilde{y}_k(p)|, \tag{69}$$

where $\widehat{y}_k(p)$ is the solution obtained from a specified discretization method and $\widetilde{y}_k(p)$ is the (assumed exact) solution calculated by direct solution the differential equation which defines the differential repetitive process under consideration. This is done by the use of the core MATLAB ODE (Ordinary Differential Equations) solvers called from a specially developed MATLAB based toolbox for the control related analysis of repetitive processes (see [30], [32] and [10] and the next two sections respectively for more details).

In the case of the 2–dimensional plots, we use (in Euclidean norm terms, i.e. $||\cdot||_2$), the quantity

$$\left\{||(y_k^{ref} - y_k^m)||_2\right\}_{k=0,1,\dots}, \tag{70}$$

where y_k^{ref} denotes the so–called reference vector, i.e. the pass profile of the differential process and y_k^m denotes the output of the resulting discrete equivalent process obtained by the use of a discretization method (denoted by m).

The first example we consider is the case when the matrices in (1) are defined by

$$\widehat{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}, \ \widehat{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \ \widehat{B}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\widehat{C} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \widehat{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \widehat{D}_0 = \begin{bmatrix} -0.1 & 0 & 0 \\ -1 & 0.6 & 0 \\ 1 & 1 & -0.1 \end{bmatrix}, \tag{71}$$

with boundary conditions

$$\begin{aligned} x_{k+1}^1(0) &= 1, \ k \geqslant 0 \\ x_{k+1}^2(0) &= 0, \ k \geqslant 0 \\ x_{k+1}^3(0) &= 1, \ k \geqslant 0 \\ y_0^1(t) &= 1, \ 0 \leqslant t \leqslant 2 \\ y_0^2(t) &= sin(2\pi t/\alpha), \ 0 \leqslant t \leqslant 2 \\ y_0^3(t) &= 0, \ 0 \leqslant t \leqslant 2, \end{aligned} \tag{72}$$

and control input sequence

$$u_k^1(t) = 1, \ k \geqslant 1, \ 0 \leqslant t \leqslant 2$$

$$\begin{aligned} u_k^2(t) &= 1,\ k \geqslant 1,\ 0 \leqslant t \leqslant 2 \\ u_k^3(t) &= 0,\ k \geqslant 1,\ 0 \leqslant t \leqslant 2. \end{aligned} \tag{73}$$

Note. The superscript in the variables on the left–hand sides of these last two equations denote the particular channel in the state, pass profile and control input vectors respectively and we continue to use this notation where appropriate in the remainder of this paper.

One key performance measure for these approximations is their numerical accuracy. For example, it has been shown elsewhere that for discrete linear repetitive processes it is possible to construct an equivalent 1D linear systems state space model of the dynamics of these processes, which (as one immediate use) can be used to study certain physically based definitions of controllability for discrete linear repetitive processes. (see [8], [13], [14], [27]).

In essence, this equivalent 1D model represents the dynamics of these processes by a 1D discrete linear systems state space model of the following form where, unlike the 1D equivalent model for other classes of 2D linear systems (see [23]), the crucial feature (in terms of its use in onward analysis) is that the vectors which define have constant dimensions and the corresponding matrices have constant entries

$$\begin{aligned} \mathbf{Y}(l+1) &= \Phi\mathbf{Y}(l) + \Delta\mathbf{U}(l) + \Theta x_l(0) \\ \mathbf{X}(l) &= \Gamma\mathbf{Y}(l) + \Sigma\mathbf{U}(l) + \Psi x_l(0), \end{aligned} \tag{74}$$

where

$$x_l(0) = d_l,\ l = 1, 2, \ldots. \tag{75}$$

Suppose, therefore, that a particular discrete approximation of the dynamics of a given differential linear repetitive process has been constructed and it is proposed to use the 1D equivalent state space model for analysis. (Such productive use will clearly require a detailed analysis of the effects of discretization on the relevant systems theoretic property of (1)). Then a possible drawback to this approach is that the dimensions of the matrices involved could be "very high". This is because when we discretize an example, the dimensions of the matrices $\Phi, \Delta, \Theta, \Gamma, \Sigma, \Psi$ in the resulting 1D equivalent model of the resulting discrete linear repetitive process will directly depend on discretization period T used. To illustrate this point, note that if we discretize the process described by (71)–(73) for $T = 0.05$ and $T = 0.01$ respectively then in the resulting 1D equivalent model of (74), the dimensions of matrices $\Phi, \Delta, \Theta, \Gamma, \Sigma, \Psi$ are (123×123) and (603×603) respectively.

4.1. Description of Figures 2 – 5

Figure 2 shows the exact solution for channel number 3 in the pass profile vector of the differential linear repetitive process described by (71)–(73) (here we use the expression "channel number 3" as a verbal description of the third entry in $y_k(p)$). A total of 26

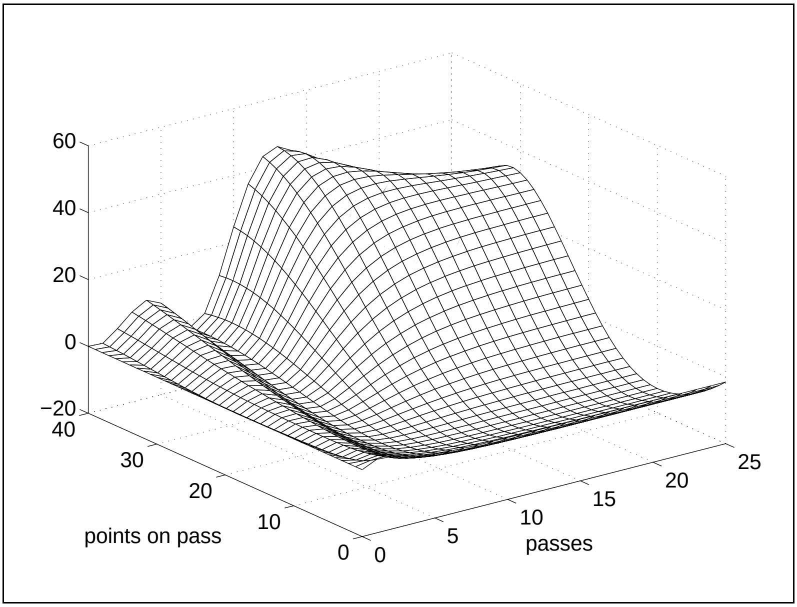


Figure 2. Exact solution for channel number 3 of the pass profile vector generated by (71)–(73).

passes (including the initial pass profile which is numbered by 0) are given in the plot, and this solution is treated as the entry $\widetilde{y}_k(p)$ in (69).

Figure 3 shows a comparison of various discretization methods (in this case for discretization period $T = 0.05$) in the form of 3–dimensional plots of absolute errors (69) for channel 3 of the pass profile vector of the process under consideration.

>From these plots, it is clear that the ZOH, forward and backward methods (detailed in Sections 3.1, 3.3, 3.4 respectively) perform much more poorly than the other methods of Section 3. Hence we conclude that these 3 methods should not be used for discretizing repetitive processes. All of the other methods, coupled with the improvements developed in Section 3, give much better results and are the recommended methods for the basic problem considered here.

Another point to note here is that 2–dimensional plots, such as those of Figures 4 and 5, do not display this critical information concerning these "bad" or inaccurate methods. Conversely, 3–dimensional plots are really only appropriate for an overall view of the trends in the response of a repetitive process. In particular, such plots do not allow us to read off the exact errors values – for this we need the more accurate 2–dimensional plots.

Figures 4 and 5 have used (70) to calculate the norms of the errors. As the reference signal y_k^{ref} the pass profile given in Figure 2 was used. Figure 4 shows the errors produced by given method with different discretization periods T as parameters, and Figure 5 shows errors under a given discretization period T by different methods and clearly the smallest errors result from using the higher order method (see Section 3.7). The next method which gives the smallest errors is the improved higher order method (see Section 3.8).

It is important to note that the improved higher order method produces practically "flat" error curves on each pass, whereas the higher order method gives very "uneven" error curves. This behaviour is clearly visible on comparing the plots in Figures 4c, 4d and 5a – 5d for the example considered here. As the period T decreases (as in Figure 5f where $T = 0.0025$) all the methods, except the trapezoidal one, exhibit very similar performance and in this case the improved trapezoidal approximation, as the simplest to implement numerically, is probably the best choice. For long discretization periods (e.g. $T = 0.2$, and $T = 0.1$ – see Figures 5a, 5b respectively), the higher order methods are preferable. This is especially important in the context of constructing 1D equivalent model which was briefly discussed in Section 4.

5. Verification of discretization results

All numerically based approximation results clearly must be verified. As discussed above, verification can be undertaken using the MATLAB ODE solvers which have been adopted as the basis for numerical work with differential linear repetitive processes (for details, see [10], [30] and [32]). The solvers are based on some very accurate adaptive methods for solving differential equations (i.e. methods where the discretization periods are selected dynamically based on local numerical conditioning). As a result, differential linear repetitive processes can be simulated numerically to any required accuracy (in practice, the only limitation could possibly be the time required for completion of the necessary calculations). Such a solution can be treated as an exact solution (factor $\widetilde{y}_k(p)$ in equation (69)).

It is important to stress the main difference between the process of numerical solution of a differential equation and the process of discretization of a continuous–time model to obtain its discrete–time counterpart. In the first case, the discretization period can be adjusted according to the local numerical conditioning of the model and if the model is numerically badly conditioned, the discretization period can be decreased. If, however, the opposite is true, the discretization period can be increased to speed up the calculation time. As a result, the mesh points are not evenly located.

In the case of the discretization procedure for a given differential linear repetitive process state space model, we must choose the discretization period in advance and then calculate the required discrete–time equivalent model (i.e. (3) and (4)). Next, the resulting discrete–time model should be simulated and the results obtained compared with the exact solution. If the resulting errors are not acceptable (in some appropriate sense), a smaller discretization period should be chosen and the procedure repeated. This process should then be repeated the required number of times to obtain the demanded accuracy.

The following question now arises: Is it possible to choose the discretization period for constructing the discrete equivalent model based on the exact numerical solution of the defining equations of a differential linear repetitive process?
In general, it is very difficult to give an exact answer to this question. A first attempt could be to choose the smallest local discretization period returned by the MATLAB ODE solver used. Then we can make the assumption that when discretizing a given differential linear repetitive process model with such a discretization period (in seconds) we will obtain the resulting discrete linear repetitive process state space model with approximately the same accuracy. In practice, our experience has been that this assumption leads to quite good results. It should be stressed, however, that in some cases this approach cannot be used as the smallest discretization period chosen by the ODE solver is so small that it has no practical relevance.

To illustrate the above approach, consider again the differential linear repetitive process described by (71)–(73). Then on directly solving the defining equations in this case, (for $k = 0, 1, ..., 25$) we obtain the statistics given in Table 1. Here we see that the smallest and longest discretization periods are $T = 1.0566 \times 10^{-7}$ and $T = 0.056399$ respectively (they are printed bold in Table 1). The first value is very small and from practical point of view it is not possible to construct the resulting discrete linear repetitive process state space model with such small discretization period. Hence we conclude that the above idea cannot be applied to this particular example.

Detailed inspection of the plots in Figure 6 reveals a very irregular placement of the mesh points at the beginning of the plots. Also the enlarged fragment of Figure 6, shows the concentration of the mesh points and the value of $T = 1.0566 \times 10^{-7}$ can be easily recognized (Figure 6f). However, if we slightly perturb the process description, e.g. the boundary conditions (72) and the input sequence (73), the final results are drastically different.

To illustrate this last point, replace the boundary conditions and input sequence for the example under consideration to (the system matrices are as before)

$$\begin{array}{rcl} x^1_{k+1}(0) & = & 0,\ k \geqslant 0 \\ x^2_{k+1}(0) & = & 0,\ k \geqslant 0 \\ x^3_{k+1}(0) & = & 0,\ k \geqslant 0 \\ y^1_0(t) & = & 0,\ 0 \leqslant t \leqslant 2 \\ y^2_0(t) & = & sin(2\pi t/\alpha),\ 0 \leqslant t \leqslant 2 \\ y^3_0(t) & = & 0,\ 0 \leqslant t \leqslant 2, \end{array} \tag{76}$$

and

$$\begin{array}{rcl} u^1_k(t) & = & 0,\ k \geqslant 1,\ 0 \leqslant t \leqslant 2 \\ u^2_k(t) & = & 0,\ k \geqslant 1,\ 0 \leqslant t \leqslant 2 \\ u^3_k(t) & = & 0,\ k \geqslant 1,\ 0 \leqslant t \leqslant 2. \end{array} \tag{77}$$

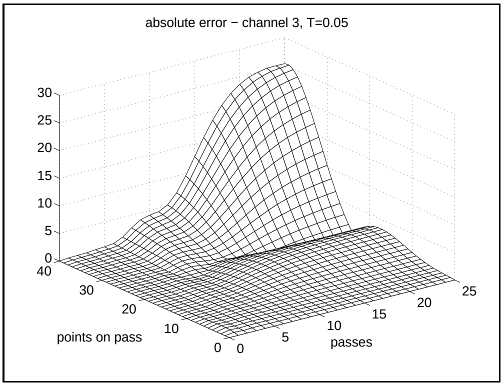
Then direct solution of the new differential linear repetitive process state space model gives the statistics of Table 2, where the shortest and the longest discretization periods are $T = 0.0023282$ and $T = 0.041126$ respectively (they are printed bold in Table 2). This solution, in contrast to the previous one, can be quite useful in practice and here the smallest discretization period has a practical meaning. In particular, detailed inspection of the results in Figure 7 reveals a slight irregular placement of the mesh points at both the beginning and at the end of the plots, but this irregularity is not as large as in Figure 6. In Figures 7a, 7b, the complete curves are given and Figures 7c–7f show the enlarged counterparts respectively.

pass no.	min. T	max. T	pass no.	min. T	max. T
1	**1.0566e-007**	0.044160	14	2.061e-007	0.032488
2	1.5849e-007	0.046186	15	2.057e-007	0.03238
3	6.3396e-007	0.051286	16	2.0546e-007	0.032596
4	1.0225e-006	0.051212	17	2.0532e-007	0.032844
5	3.9376e-007	**0.056399**	18	2.0523e-007	0.039174
6	2.8787e-007	0.053819	19	2.0518e-007	0.033189
7	2.4786e-007	0.051949	20	2.0515e-007	0.033247
8	2.2878e-007	0.046634	21	2.0513e-007	0.033265
9	2.1868e-007	0.044147	22	2.0512e-007	0.035585
10	2.1304e-007	0.039722	23	2.0511e-007	0.033249
11	2.0979e-007	0.041464	24	2.0511e-007	0.033233
12	2.0789e-007	0.038090	25	2.0511e-007	0.033742
13	2.0677e-007	0.037715			

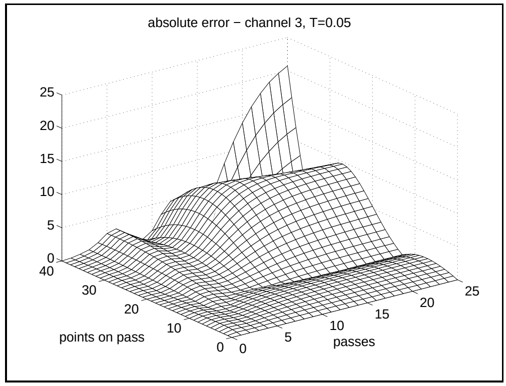
Table 1. Statistics of the numerical solution of the process described by (71)–(73).

pass no.	min. T	max. T	pass no.	min. T	max. T
1	0.0042412	0.031965	14	0.0029983	0.023200
2	0.0042405	0.037218	15	0.0031696	0.024512
3	0.0042406	0.031985	16	0.0052475	0.023731
4	0.0042421	0.034013	17	0.0062074	0.024763
5	0.0042452	0.035007	18	0.0074497	0.025614
6	0.0026951	0.032371	19	0.0027985	0.020193
7	0.0041733	**0.041126**	20	0.0046194	0.020061
8	0.0051301	0.035603	21	0.0059278	0.026259
9	0.0057663	0.034873	22	**0.0023282**	0.018170
10	0.0032139	0.033368	23	0.0071430	0.025967
11	0.0031260	0.027997	24	0.0102280	0.022779
12	0.0030852	0.029754	25	0.0046091	0.024328
13	0.0030383	0.028869			

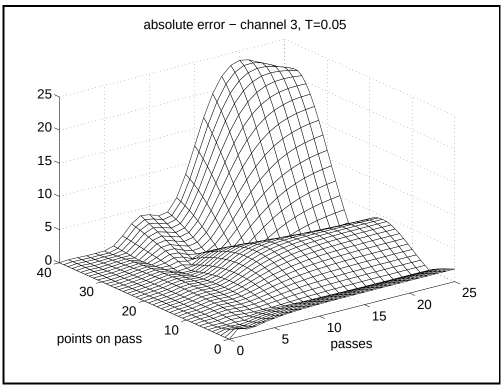
Table 2. Statistics of the numerical solution of the process described by (71), (76), (77).



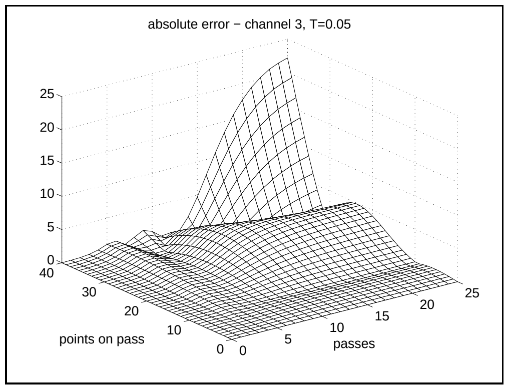
(a)
forward method



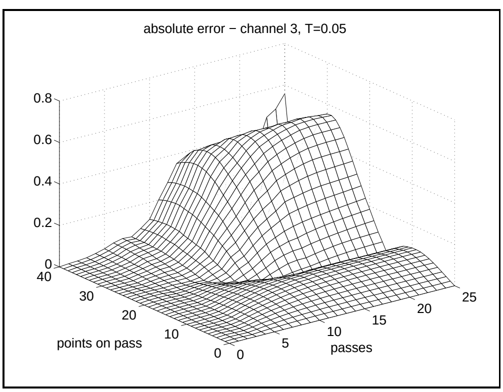
(b)
backward method (described by (27))



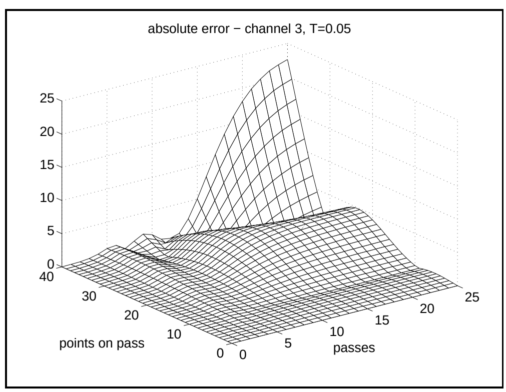
(c)
backward method (described by (38))



(d)
ZOH method



(e)
improved ZOH method



(f)
trapezoidal method

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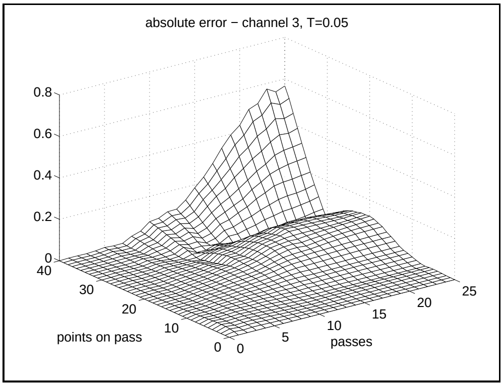
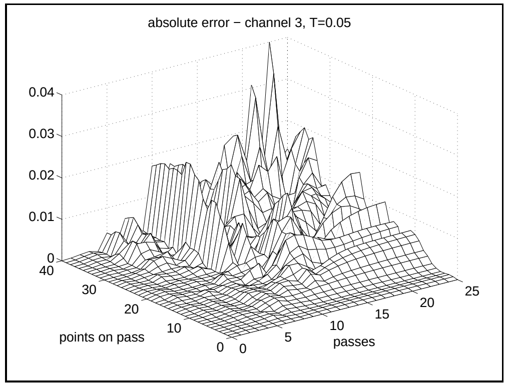
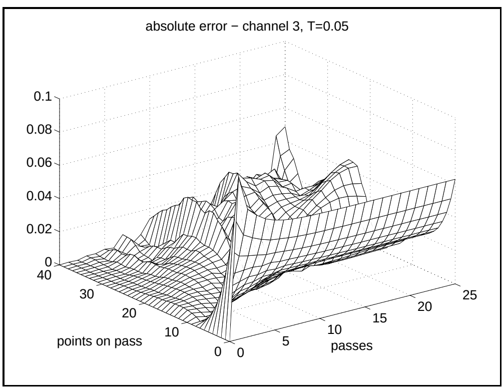


(g)
improved trapezoidal method



(h)
single step, higher order



(i)
improved single step, higher order

Figure 3. Comparison of nine different discretization methods. Figures (a)–(i) show the absolute error in the same channel between the exact solution and the given method with $T = 0.05$.

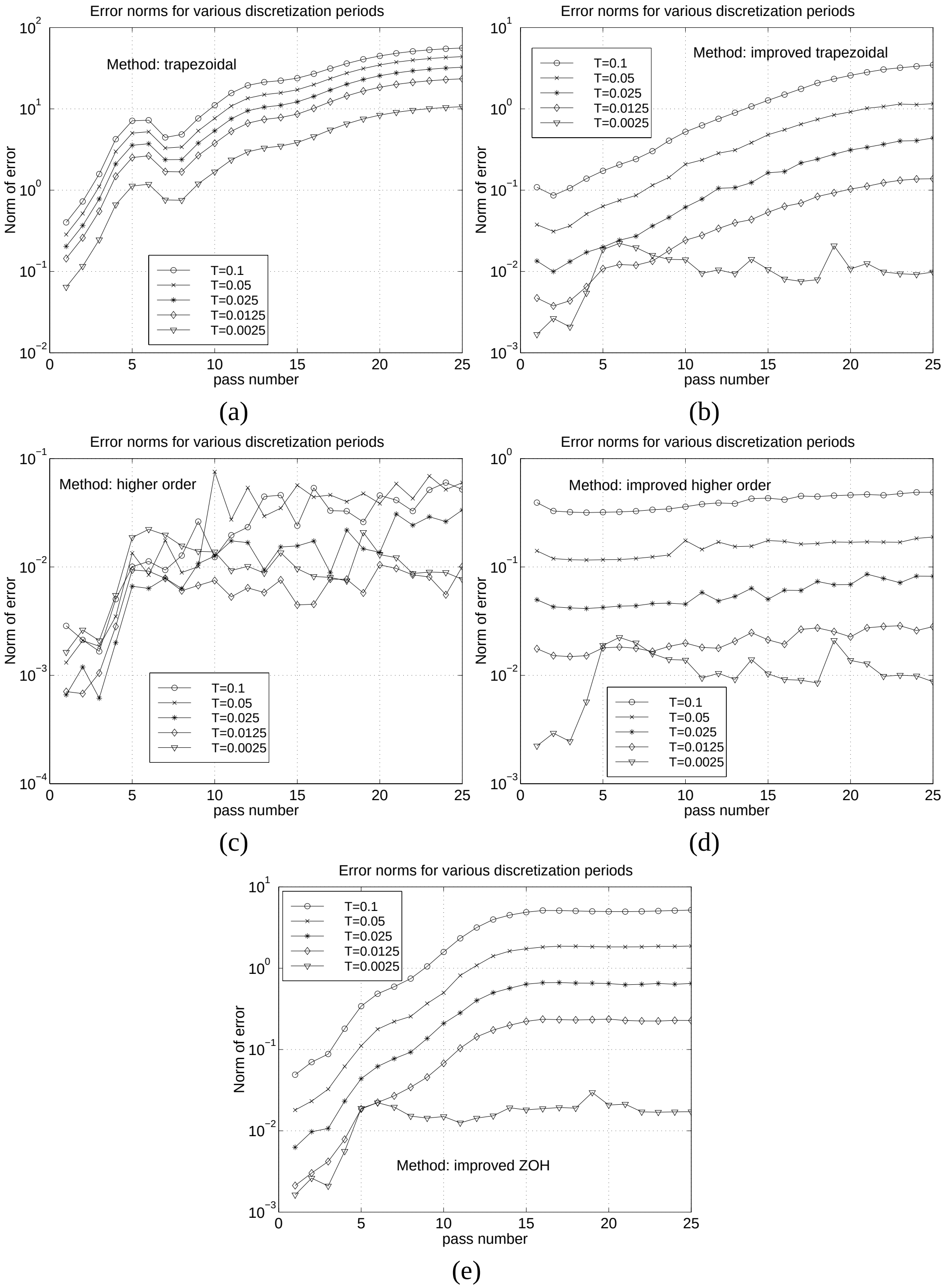


Figure 4. Error for a given discretization method and different discretization periods T for channel number 3 of the pass profile vector.

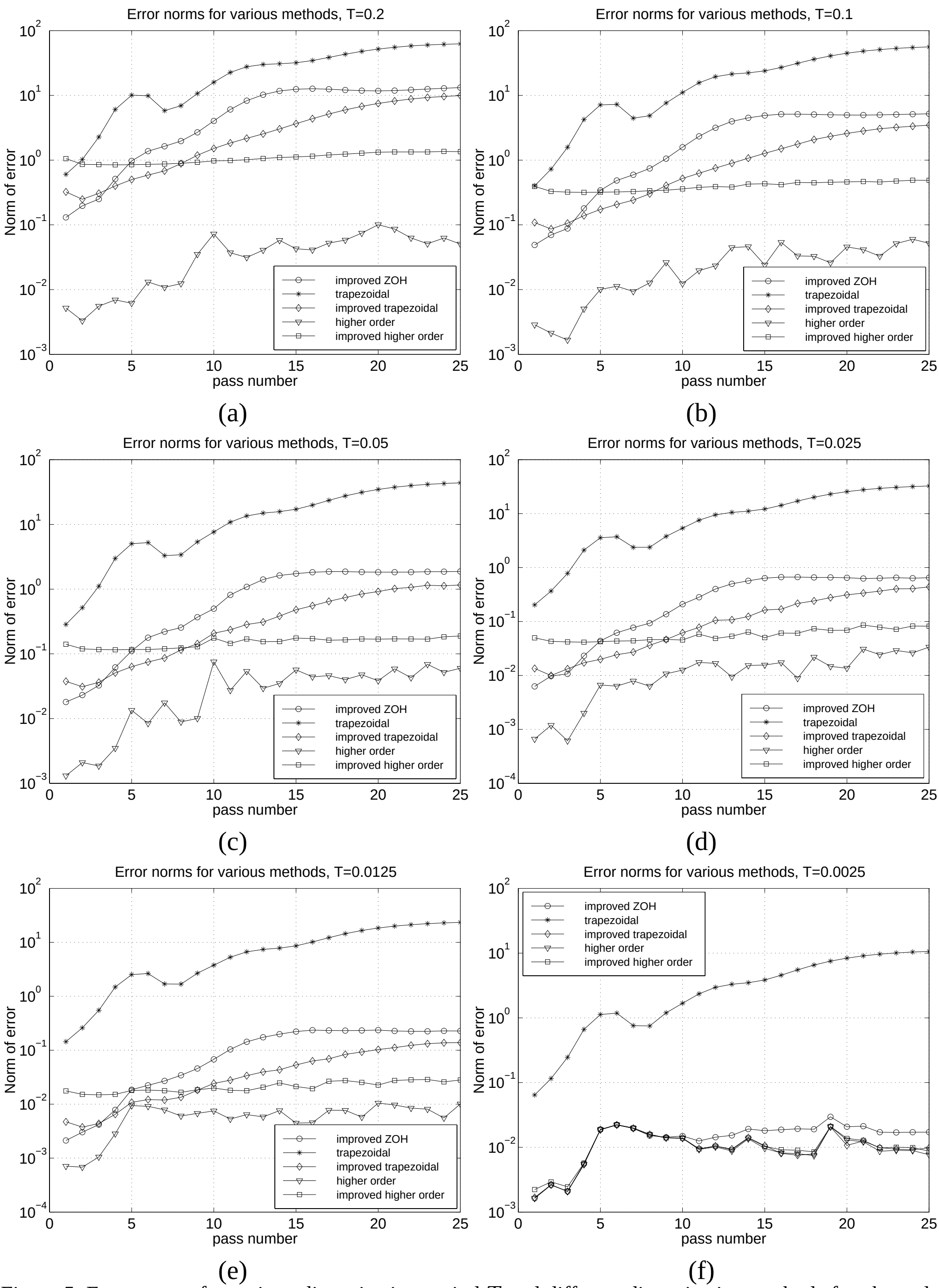


Figure 5. Error norms for a given discretization period T and different discretization methods for channel number 3 of the pass profile vector.

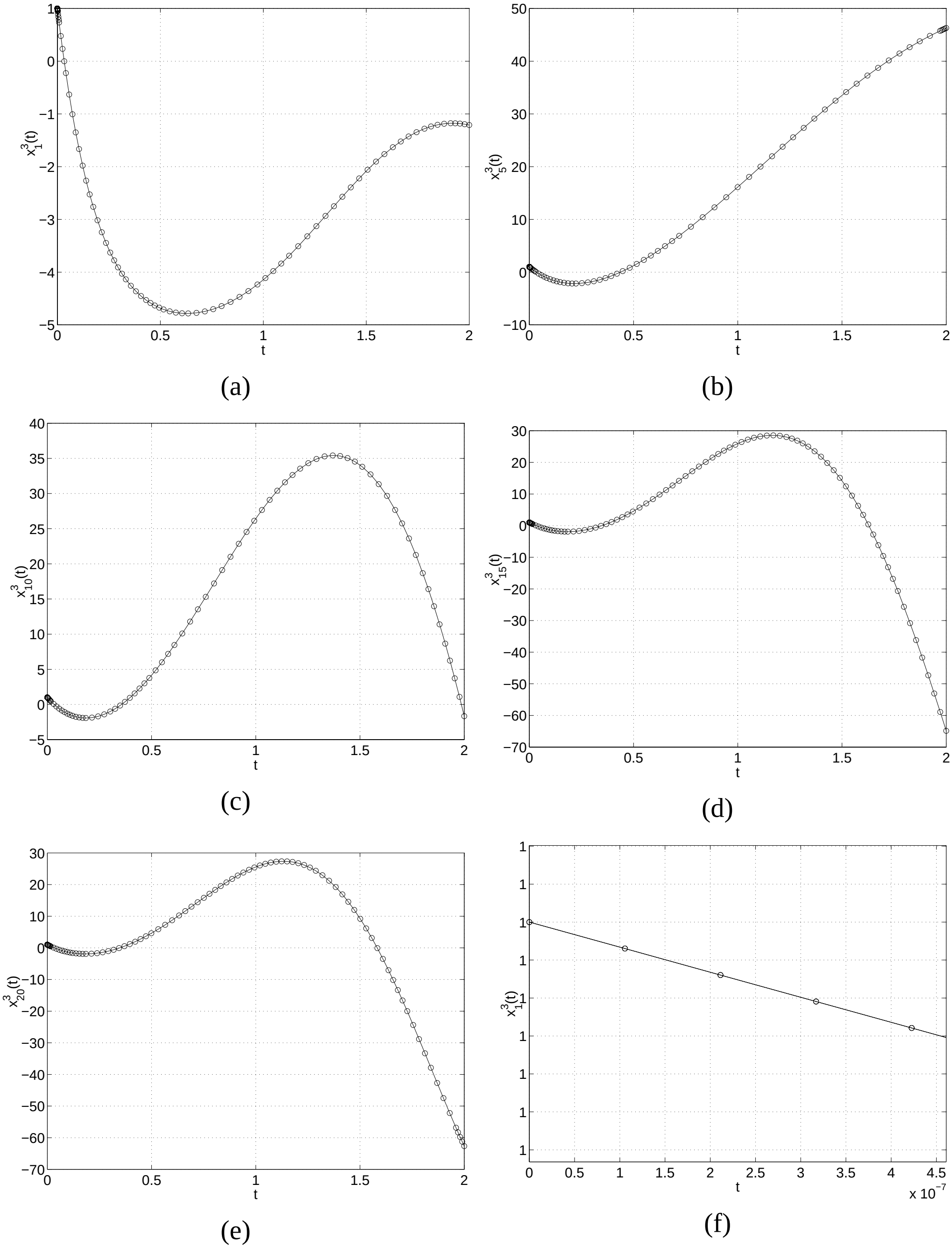


Figure 6. Placement of the mesh points during the numerical solution of the differential linear repetitive process described by (71)–(73).

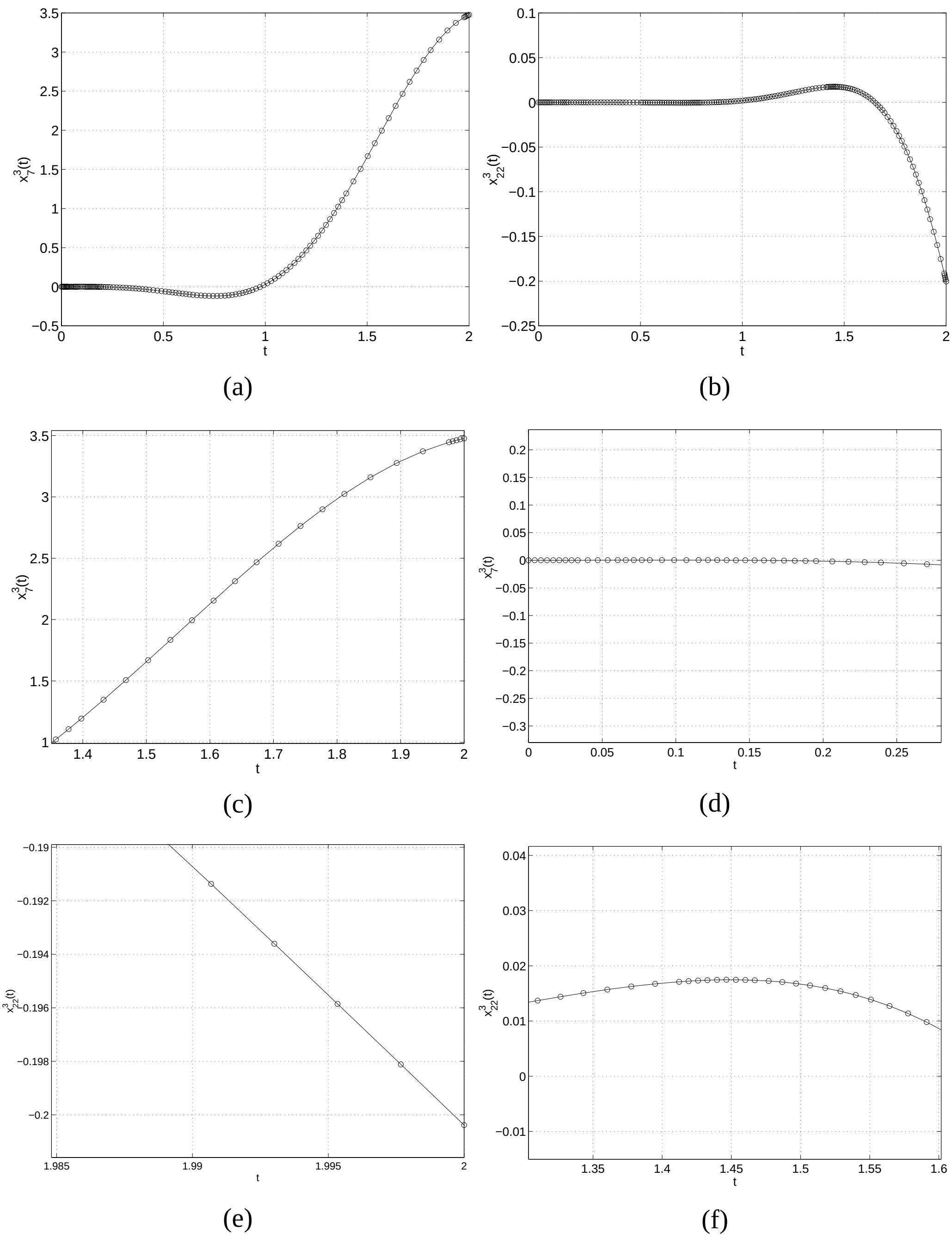


Figure 7. Placement of the mesh points during the numerical solution of the differential linear repetitive process described by (71), (76) and (77).

6. MATLAB based toolbox

In practical applications it is essential to have possibility of fast analysis (in various aspects) of a given repetitive process. This includes such tasks as:

1. convenient data input facilities of all model matrices, control inputs and initial/boundary conditions,

2. construction of discrete approximations of the dynamics of the specified differential process,

3. checking its basic system properties (e.g. stability),

4. numerical simulation of the process dynamics together with appropriate graphical display facilities for the results.

The development of a MATLAB based toolbox for linear repetitive processes is on–going and in this section we detail some basic features of its construction and illustrate some of its current features.

6.1. General Features

The core of this toolbox provides the following classes of functions:

1. Generally Applicable Functions – essentially all the necessary inputs (e.g. control inputs, initial conditions, initial pass profile, the matrices defining a state space model, discretization period, etc.) are prepared manually by the user before implementation using a given function. The implementable functions are, of course, "very similar" to those found in standard MATLAB toolboxes.

2. A User Friendly Graphical Tool – this has been designed to run within MATLAB. During operation, it is possible, for example, to modify parameters in the model being simulated and view 2D and 3D plots of, say, the resulting sequence of pass profiles.

6.2. Toolbox - Basics and Applications

Figure 8 shows the initial window of the toolbox which, in addition to data input facilities, is also the starting point for all other windows (modules) of the toolbox.

It is possible to select system inputs, states and initial pass profiles either from some predefined sets (for example a step input is represented by "111–111–111" or a "sine wave" as an initial pass profile) or any other user prepared values from the MATLAB workspace (checkbox "User defined"). Also (in addition to the most simple form) it is possible to simulate the response of a given example with so–called dynamic boundary conditions [7] (checkbox "Use ext. init. cond.").

One of the major features of the toolbox in its current form are routines for constructing the discrete approximation of a differential linear repetitive processes that produce their state space model matrices and the form of the boundary conditions (this feature has not been used in this paper). Examples of its use in constructing discrete approximations of a given differential process have been given earlier in this paper.

The toolbox contains a number of methods for displaying the response of a (differential or discrete) process to a given input sequence, states and boundary conditions. Figures 9a, 9b illustrate the 3D and 2D options for an example. In the 3D case, all passes computed are displayed on one figure (which can become "rather uninformative" if the number of passes is "large".) It is also possible to interactively edit these 3D and 2D plots, e.g. rotate it in any direction, display a selected set of pass profiles or points on a pass, etc. In the 2D view option, only a selected number of passes can be displayed.

7. Conclusions

In the paper various methods for constructing discrete approximations of the dynamics of differential linear repetitive processes have been detailed. In such processes only one of two independent variables is continuous and this may suggest that all that is required here is direct application of discretization methods known from 1D systems analysis. This statement is incorrect and in actual fact the clear 2D systems structure of

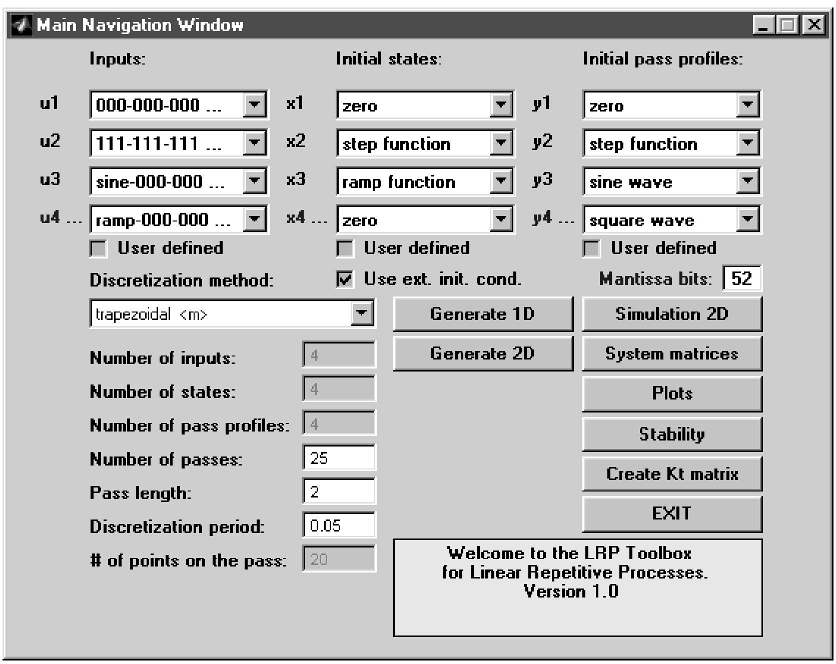
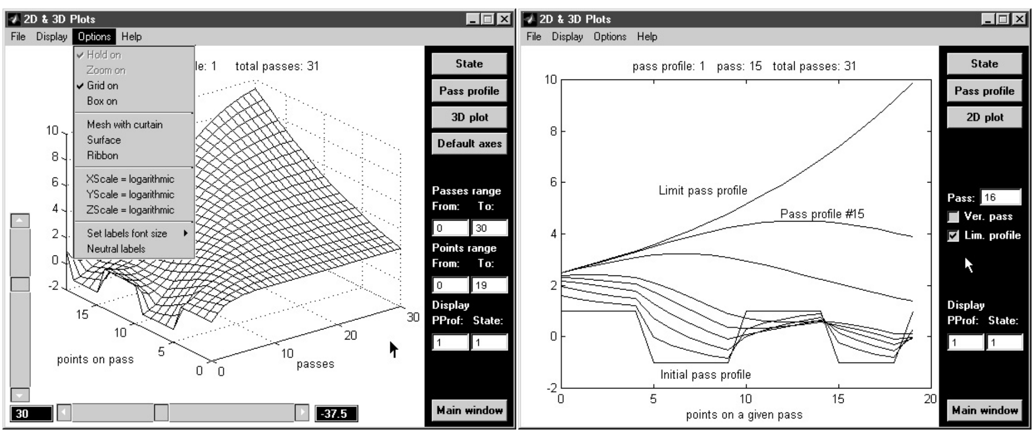


Figure 8. The main navigation window of the MATLAB toolbox.

repetitive processes leads to the need to either develop new analysis tools or, as is the case with the construction of discrete approximations which is the subject of this paper, undertake non-trivial extensions of currently available tools. Here a novel improvement of classical trapezoidal and Zero–Order–Hold approximation methods has been developed. Also a higher order method, not commonly used in 1D systems, has been applied to this problem and an improved version of it is developed.

All methods considered/developed here have been numerically compared. From these studies, we conclude that a higher order method gives the smallest errors but its more complicated (in relative terms) structure means that it is also the most demanding in terms of the computation time required. Its improved version gives slightly worse results, but the discretization errors are practically "flat" on each pass. An improved trapezoidal method can be seen compromise between relatively small discretization errors and computational simplicity. As the period T decreases all methods of practical importance give essentially the same accuracy and since an improved trapezoidal approximation is numerically the simplest to implement, this is the best choice under this measure of relative performance.

As a final point, note again that in most cases a stable (asymptotic or along the pass) differential example does not necessarily lead to a discrete linear repetitive process state space model with the same stability properties, although in all cases it is possible to write down conditions under which the resulting discrete process is stable (asymptotic and along the pass).This strongly suggests that there will be non-trivial problems to solve when considering the effects of the approximation process on key systems theoretic properties of differential linear repetitive processes, e.g. when considering digital



(a) (b)

Figure 9. (a) 3D plots example windows, (b) 2D plots example windows.

implementation of (suitably structured) control laws. Progress in these and other areas ill be reported in due course.

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