Archives of Control Sciences
Volume 12(XLVIII), 2002
No. 1–2, *pages 125–149*

Feed-forward torques and reference trajectory for an arm with flexible joints

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We propose a design method of the feed-forward torques and the reference trajectory for an arm with flexible joints and unknown stiffness coefficients. The bounds on the control torque are included explicitly. The designed commanded feed-forward torques and the corresponding reference trajectory are close to be time-optimal. The control law for each drive torque is composed with the commanded feed-forward torque and linear angular position and velocity feedback. The torques and the trajectory are used as input signals of PD-controllers.

In a case of one-link flexible arm it is proved that so-called 'fluent' control torque enables to reach the desired motion with a small error and without large vibrations in the flexible joint. Designed fluent commanded feed-forward torque is successfully implemented in numerical experiments. The stiffness coefficient is assumed to be unknown so the mathematical model of the flexible arm is not known exactly.

In two-link flexible arm the stiffness coefficients are also assumed unknown. Due to discontinuities of the optimal control functions and not acceptable jumps in the optimal control torques a 'trapezoidal' fluent control technique is proposed. Numerical experiments show that the designed control is close to time-optimal.

The approach presented in the paper can be extended to systems with more links and can take the gravity into consideration. However, in this case a time-optimal or a quasi-time-optimal control for the associated rigid system have to be designed.

Key words: one–two–link arm, flexible joint, control torque, torque bound, reference trajectory, feed-forward and feedback control

1. Introduction

An arm with one or two rigid links and with flexible joints is considering in this paper. We take into account the bounds on the control torques explicitly. An arm with one rigid link and one elastic joint is a mechanical system of two degrees of freedom. The motion equations of this system are of fourth order. Two-link arm with two elastic joints has four degrees of freedom - two rigid and two flexible degrees of freedom. The mathematical model of this system is of eighth order. It is not so easy to study a model

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of such high order. For example, it is difficult to design the time-optimal control for the system of high order. It is impossible to design the time-optimal control, if the system parameters are unknown. However, the stiffness coefficients for example are often known only approximately. Thus, the mathematical model of a flexible arm is unknown. But the requirement for the control system to obtain a fast motion is important and the optimal control is sensitive with respect to the parameter values. We assume here that all parameters of the associated rigid arm, without elasticity in the joints, are well known i.e., the model of the associated rigid arm is well known. Therefore, at first we design a time-optimal or a quasi-time-optimal control for the associated rigid arm with one or two links. After we adjust this control to the arm with elastic joints to ensure fast motion and small elastic vibrations simultaneously. A time-optimal control with limited amplitude is discontinuous usually [26]. Our time-optimal or quasi-time-optimal control for an associated rigid arm has discontinuities as well. If we apply this control to the arm with elastic joints, unwanted vibrations appear in these joints, because the switching instants for an associated rigid arm are not suitable for an elastic arm. It takes time to damp these vibrations. Therefore, the designed discontinuous control is modified in the area of the switching times to decrease the influence of the discontinuities. The discontinuous control function is replaced near the switching by a piecewise linear continuous function. In this way we compute a new control function - fluent control. The inclination of the linear parts depends on the joint elasticity. If the stiffness coefficient is 'small', this inclination has to be 'small' as well. Thus, we can adapt the inclination to any elasticity. Under a chosen fluent control, little elastic oscillations appear in the joints and the motion of the flexible arm is close to the motion of an associated rigid arm. We employ fluent control torque as a feed-forward control torque. The corresponding trajectory of an associated rigid arm is used as a reference trajectory. The control law for the flexible arm contains a feed-forward signal and a linear position and velocity feedback (usual PD-controller). It means that the designed quasi-time-optimal control torques and the corresponding trajectory of associated rigid arm are used as input signals for PD-controllers. Designed in this way, the control law does not excite large elastic vibrations. What is important: no information is necessary about the deformation in the elastic joints to control the flexible arm.

The flexibility in the joints is more important than a flexibility of the links for many manipulators. Therefore, many investigators have studied the problem of control of an arm with rigid links and elastic joints [1, 3, 6–8, 10, 11, 19, 22, 23, 25, 28–30]. First articles devoted to this problem were published long time ago. However the last years there are publications concerning this topic too (see for example, [12–14, 20, 21, 24, 27]). In works [4, 5], we applied our method to control an arm with one and two flexible links. Experiments have confirmed the efficiency of this approach for the arms with flexible links. Here this approach is developed for an arm with rigid links, but with flexible joints. To our knowledge, this approach is new. Unlike many other authors, we consider here limited control torques and the time-optimal control problem.

Thus, the objective of this paper is to design such a control in order to avoid the exciting of big elastic vibrations. On the other hand, the motion of the manipulator under this control has to be fast. These two requirements are contradictory; therefore we must search for a compromise.

The paper is organized as follows. Section 2 is devoted to one-link arm with flexible joint. Well-known model of this arm is recalled in subsection 2.1. The problem is stated in subsection 2.2. We compare the bang-bang and the fluent controls in subsection 2.3 analytically. In subsection 2.4 the control law is designed and numerical experiment with a one-link flexible arm under this control is described. Section 3 is devoted to a two-link manipulator with two flexible joints. At first in subsection 3.1 the mathematical model of the two-link arm with flexible joints is described. The problem for this arm is stated in subsection 3.2. The mathematical model of an associated rigid two–link arm is recalled in subsection 3.3. Quasi-time-optimal control for this rigid manipulator is described in subsection 3.4. In subsection 3.5 a fluent control and the corresponding trajectory are described. The control law for an arm with elastic joints and the numerical experiment under this control is presented in subsection 3.6. Section 4 contains conclusion.

2. One-Link arm with flexible joint

Here the simple case of a one–link manipulator with elastic joint is studied.

2.1. Mathematical model

Let us consider an actuated rigid link of length L. The link can be rotated on the horizontal plane around motionless point O. We denote by j its moment of inertia about this end point O. The other link end is clamped to the mass center of a payload. We denote by M and J the mass of this load and its moment of inertia about the mass center, respectively. A motor actuates the motion of the link. The rotor of the actuator is modeled as a symmetrical relative to the rotation axis body. Let J_m be the inertia moment of this rotor about its axis, and Γ be the control torque. Let us take into consideration an elasticity of the transmission between the motor armature axe and the link hub. This elasticity is modeled as a torsional linear spring located after the gear; n is the gear ratio, k is the stiffness coefficient of the spring.

We denote by θ and θ_m articulated coordinates, respectively, of the link and of the motor rotor (Fig. 1). The mathematical model of this system with two degrees of freedom is well known [28]:

$$\begin{aligned} a\ddot{\theta} + k\left(\theta - \tfrac{\theta_m}{n}\right) &= 0 \\ J_m\ddot{\theta}_m - \tfrac{k}{n}\left(\theta - \tfrac{\theta_m}{n}\right) &= \Gamma \end{aligned} \tag{1}$$

Here, value $a = j + ML^2 + J$ is the moment of inertia about point O of the rigid link with the load. We can introduce two new variables δ and ϑ instead of θ and θ_m, respectively:

$$\delta = \theta - \frac{\theta_m}{n}, \qquad \vartheta = \theta + \frac{J_m n}{a}\theta_m \tag{2}$$

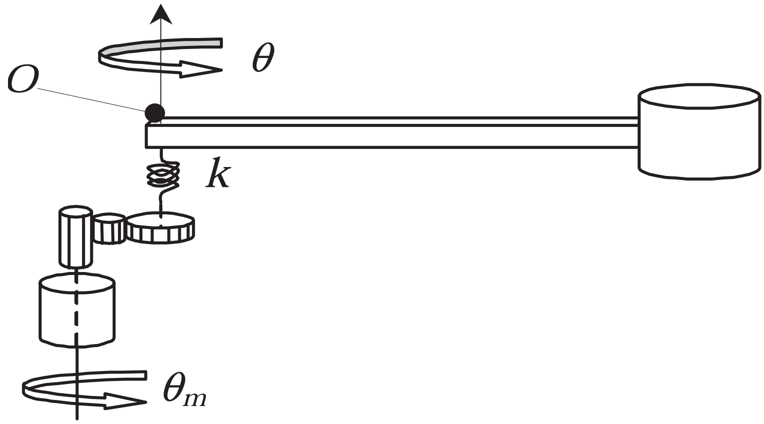

Figure 1. Scheme of the link with flexible joint

These two variables are governed by the following equations (they follow from (1)):

$$\ddot{\delta} + \omega^2\delta = -\frac{\Gamma}{J_m n} \qquad \left(\omega^2 = k\frac{a + J_m n^2}{aJ_m n^2}\right), \qquad a\ddot{\vartheta} = n\Gamma \tag{3}$$

Note that $a\dot{\vartheta}/n$ is the angular momentum of the arm about point O.

The control torque Γ of the motor at joint A is bounded by constant Γ_M:

$$|\Gamma| \leqslant \Gamma_M. \tag{4}$$

Later, sometimes we use variables θ and δ instead of θ and θ_m. In this case, we suppose that variable δ is substituted in equations (1) instead of variable θ_m. It follows from the first equation of (1) that deflection (deformation) δ is always in opposite direction to acceleration $\ddot{\theta}$, which is clear from a physical point of view.

2.2. Statement of the problem

The problem is to find the admissible (satisfying inequality (4)) control Γ that transfers the manipulator from its initial state

$$\theta(0) = \dot{\theta}(0) = 0 \tag{5}$$

to an arbitrary final state

$$\theta(T) = \theta_d, \qquad \dot{\theta}(T) = 0 \tag{6}$$

and keeps it in this state. Here θ_d is the desired angle of the arm rotation. The initial elastic joint deformation δ (the flexible variable) and its time derivative $\dot{\delta}$ are zero; the final values are desired to be zero too:

$$\delta(0)=\dot{\delta}(0)=0, \quad \delta(T)=\dot{\delta}(T)=0, \quad (\theta_m(0)=\dot{\theta}_m(0)=0, \quad \theta_m(T)=n\theta_d, \quad \dot{\theta}_m(T)=0) \tag{7}$$

Time T is not given, but it is required that this time should be as short as possible.

From (5) – (7) we obtain the following boundary conditions for rigid variable ϑ (see (2)):

$$\vartheta(0) = \dot{\vartheta}(0) = 0, \qquad \vartheta(T) = \left(1 + \frac{J_m n^2}{a}\right)\theta_d = \vartheta_d, \qquad \dot{\vartheta}(T) = 0 \tag{8}$$

If deformation δ in the joint does not change, then $\ddot{\delta} = \ddot{\theta} - \ddot{\theta}_m/n = 0$, and the arm moves as a rigid body. The following relations describe the dynamic behavior of this associated rigid one-link arm:

$$b\ddot{\theta} = n\Gamma, \qquad |\Gamma| \leqslant \Gamma_M \tag{9}$$

Here, value $b = a + J_m n^2$ is the moment of inertia about point O of the rigid arm with the motor armature and the load, θ is the angular coordinate of the arm.

Let us consider for system (9) boundary conditions (5) and (6). It is well known [26], that if $\theta_d > 0$, then the time-optimal control for this boundary value problem can be written in the following form (see Fig. 2):

$$\Gamma(t) = \begin{cases} \Gamma_M, & \text{if} \quad 0 \leqslant t < T/2 \\ -\Gamma_M, & \text{if} \quad T/2 \leqslant t \leqslant T \\ 0, & \text{if} \quad t < 0 \text{ or } t > T \end{cases} \tag{10}$$

Under control (10) acceleration $\ddot{\theta}$ changes according to the expression:

$$\ddot{\theta}(t) = \begin{cases} \ddot{\theta}_s, & \text{if} \quad 0 \leqslant t < T/2 \\ -\ddot{\theta}_s, & \text{if} \quad T/2 \leqslant t \leqslant T \\ 0, & \text{if} \quad t < 0 \text{ or } t > T \end{cases} \tag{11}$$

with

$$\ddot{\theta}_s = \frac{n\Gamma_M}{b} = \frac{n\Gamma_M}{a + J_m n^2} \tag{12}$$

The final minimal time with control (10) is [26]:

$$T = 2\left(\frac{b\theta_d}{n\Gamma_M}\right)^{1/2} \tag{13}$$

Function (10) describes the time-optimal control also for the second equation (3) with boundary conditions (8). Expression (13) describes minimal possible time T_s for this boundary value problem:

$$T_s = 2\left(\frac{a\vartheta_d}{n\Gamma_M}\right)^{1/2} = 2\left[\frac{(a + J_m n^2)\theta_d}{n\Gamma_M}\right]^{1/2} = 2\left(\frac{b\theta_d}{n\Gamma_M}\right)^{1/2} = T.$$

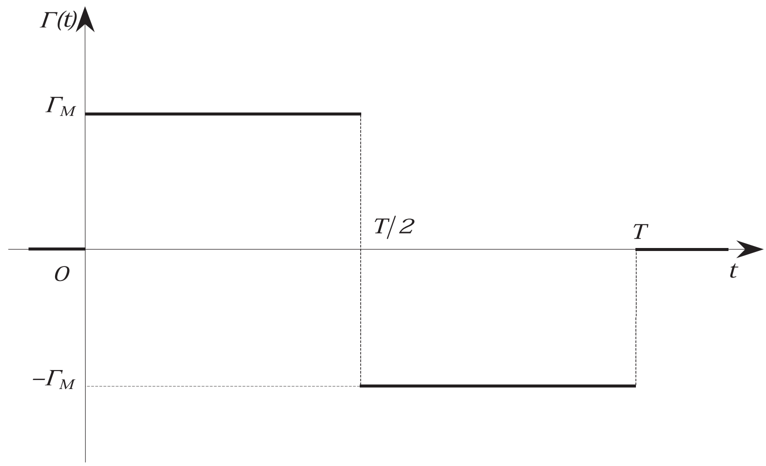

Figure 2. Time-optimal control for the system (9)

The minimal possible time to that the system (3) satisfies both boundary conditions (7) and (8) is not smaller than the value (13). It means that minimal time (13) for the rigid arm is the low estimation of the minimal possible time for the flexible joint arm. In general, the minimal possible time for the flexible joint arm is larger than for the associated rigid arm. Pontryagin’s maximum principle [26] can be used to design the time-optimal control for system (1) or (3), if all parameters of this system are well known. In general, the number of switching points in the time-optimal control for this system is not one as for system (9), but larger. However, usually the stiffness coefficient k is known only approximately. Therefore, we try to adjust time-optimal control (10) to an arm with the elastic joint. Function (10) contains jumps. In subsection 2.3, it is shown that these jumps are not acceptable for an arm with flexible joint because large elastic vibrations in the joint appear. We modify this function and use the ‘fluent’ [2, 4, 5] control.

2.3. Analytical comparison of bang-bang and fluent controls

Time-optimal control (10) is a bang-bang function with one switching point at the middle of the motion. At the start, $t = 0$, of the controlling process and at its end, $t = T$, this optimal control has discontinuities as well. Let us consider the initial part of this control with one switching point (see Fig. 3):

$$\Gamma(t) = \begin{cases} 0, & \text{if} \quad t < 0, \\ \Gamma_M, & \text{if} \quad t \geqslant 0, \end{cases} \tag{14}$$

Under control torque (14), equations (1) (see also system (3)) has the following stationary solution:

$$\delta_s = -\frac{\Gamma_M}{\omega^2 J_m n} = -\frac{an\Gamma_M}{k(a + J_m n^2)}, \qquad \ddot{\theta}_s = -\frac{k}{a}\delta_s = \frac{k\Gamma_M}{a\omega^2 J_m n} = \frac{n\Gamma_M}{a + J_m n^2} \tag{15}$$

The second expression (15) coincides with (12).

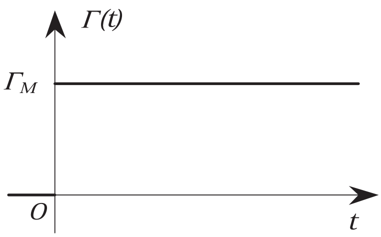

Figure 3. Discontinuous control

With initial conditions (7), the solution of equations (1) under control (14) is:

$$\delta(t) = \delta_s(1-\cos\omega t), \qquad \ddot{\theta}(t) = -\frac{k}{a}\delta(t) = \ddot{\theta}_s(1-\cos\omega t). \tag{16}$$

The oscillations of system (1) with control (14) around stationary solution (15) are described by formulas (16).

Consider now, instead of control (14), another control, which changes continuously (see Fig. 4):

$$\Gamma(t) = \begin{cases} 0, & \text{if} \quad t < 0, \\ \frac{\Gamma_M}{\tau} t, & \text{if} \quad 0 \leqslant t < \tau, \\ \Gamma_M, & \text{if} \quad t \geqslant \tau. \end{cases} \tag{17}$$

With initial conditions (7), under 'fluent' control (17), we obtain the following solution

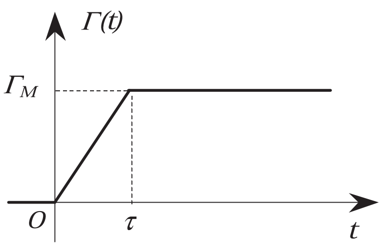

Figure 4. Piecewise linear control function

of equations (1):

$$\delta(t) = \begin{cases} \delta_s\left(\frac{t}{\tau} - \frac{\sin\omega t}{\omega\tau}\right), & \text{if} \quad 0 \leqslant t < \tau, \\ \delta_s\left[1 - \frac{2}{\omega\tau}\sin\frac{\omega\tau}{2}\cos\omega\left(t-\frac{\tau}{2}\right)\right], & \text{if} \quad \tau \leqslant t, \end{cases} \tag{18}$$

$$\ddot{\theta}(t) = \begin{cases} \ddot{\theta}_s \left(\frac{t}{\tau} - \frac{\sin \omega t}{\omega \tau} \right), & \text{if} \quad 0 \leqslant t < \tau, \\ \\ \ddot{\theta}_s \left[1 - \frac{2}{\omega \tau} \sin \frac{\omega \tau}{2} \cos \omega \left(t - \frac{\tau}{2} \right) \right], & \text{if} \quad \tau \leqslant t, \end{cases}$$

Formulas (18) show that, if value τ increases, then the amplitudes of the vibrations of deflection δ and acceleration $\ddot{\theta}$ around the piecewise linear functions of time

$$\delta(t) = \begin{cases} \delta_s \frac{t}{\tau}, & \text{if} \quad 0 \leqslant t \leqslant \tau, \\ \\ \delta_s, & \text{if} \quad \tau \leqslant t \end{cases}, \qquad \ddot{\theta}(t) = \begin{cases} \ddot{\theta}_s \frac{t}{\tau}, & \text{if} \quad 0 \leqslant t \leqslant \tau, \\ \\ \ddot{\theta}_s, & \text{if} \quad \tau \leqslant t \end{cases} \tag{19}$$

decrease. The vibrations for time $t > \tau$ can be vanished, if we choose $\tau = 2\pi/\omega$ or $\tau = 4\pi/\omega \ldots$ (see the second lines in formulas (18)). However, if stiffness coefficient k is not well known, the eigen frequency ω is not well known too and we can not choose time τ in this way. It is possible, by the choice of time τ, to make the vibrations around motion (19) small, as desired (it is clear from a physical point of view as well). It is especially relevant for time $t > \tau$. But if time τ is large, motion (19) is not admissible because acceleration $\ddot{\theta}$ goes to its maximal value $\ddot{\theta}_s$ with a long time and we deviate from the time-optimality. Note however, that expressions (18) contain product $\omega\tau$ in the denominator, at the same time the module of the numerator is bounded above. Therefore, if product $\omega\tau$ is 'large', the oscillation amplitude is 'small'. In real manipulators, the stiffness in the joints and consequently eigen frequencies are 'large' usually. But if frequency ω is large enough, then we can obtain large product $\omega\tau$, choosing small time τ. In that case, the deviation from the time-optimality will be small. However for each concrete system with flexible joints, it is necessary to estimate, if the deviation from the time-optimality is small or not. For such estimation, we can use computations, described in the next sections.

Thus, 'fluent' control (17) enables the manipulator to avoid the 'large' elastic vibrations of the arm. Therefore, we use the piecewise linear approximation for the synthesis of the control for a flexible one-link arm. For $\theta_d > 0$ we design desired nominal torque $\Gamma_d(t)$ in the following 'trapezoidal' [4, 5, 9] form (see Fig. 5):

$$\Gamma_d(t) = \begin{cases} \frac{\Gamma_M}{\tau} t, & \text{if} \quad 0 \leqslant t < \tau, \\ \Gamma_M, & \text{if} \quad \tau \leqslant t < T/2 - \tau, \\ \frac{\Gamma_M}{\tau}(T/2 - t), & \text{if} \quad T/2 - \tau \leqslant t < T/2, \\ -\Gamma_d(T - t), & \text{if} \quad T/2 \leqslant t \leqslant T, \\ 0, & \text{if} \quad t < 0 \text{ or } t > T. \end{cases} \tag{20}$$

In (20), T is some new value (not (13)), which we need to calculate. Function (20) is symmetric in interval $[0, T]$ with respect to point $(T/2, 0)$: $\Gamma_d(t) \equiv -\Gamma_d(T - t)$.

If we give time τ, then equation (9), control (20), boundary conditions (5) and (6) form the boundary value problem. It is easy to solve this problem numerically by iterations. Solving this problem we obtain time T, the corresponding trapezoidal torque, the position and velocity as functions of time. During numerical experiment, with flexible arm, these computed functions are used as commanded values $\Gamma_d(t)$, $\theta_d(t)$, $\dot{\theta}_d(t)$ (see subsection 2.4).

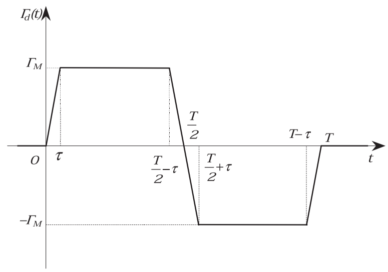

Figure 5. Trapezoidal control function

2.4. Numerical experiment with flexible one-link arm

During the motion, computer calculates commanded values $\Gamma_d(t)$, $\theta_d(t)$, $\dot\theta_d(t)$ (as it is written above) and control torque Γ, which consists of feed-forward torque $\Gamma_d(t)$ combined with a linear position and velocity feedback [10, 17, 28–30]:

$$\Gamma = \Gamma_d(t) + \beta[n\theta_d(t) - \theta_m] + \gamma[n\dot\theta_d(t) - \dot\theta_m] \tag{21}$$

Expression (21) means that we use calculated torque $\Gamma_d(t)$ and functions $\theta_d(t)$, $\dot\theta_d(t)$ as input signals for the usual PD-controller. To realize control law (21) we have to measure the angle of armature rotation θ_m and angular velocity $\dot\theta_m$. But it is not necessary to measure deformation δ.

For the numerical experiments, we use the following parameters of the arm with homogeneous link:

$$\begin{gathered} L = 1 \text{ m}, \quad J_m = 1.8 \cdot 10^{-2} \text{ kg} \cdot \text{m}^2, \quad n = 5, \quad k = 100 \text{ N} \cdot \text{m}, \\ j = 0.6864 \text{ kg} \cdot \text{m}^2, \quad M = 7.29 \text{ kg}, \quad \Gamma_M = 0.75 \text{ N} \cdot \text{m} \end{gathered} \tag{22}$$

In control (21), as well as feed-forward torque $\Gamma_d(t)$ there are terms with feedback signals too. Therefore, calculating feed-forward torque $\Gamma_d(t)$ we have to decrease slightly limit Γ_M for it in order to satisfy inequality (4) with control (21). If we use in our numerical experiments limit $\Gamma_M = 0.6$ N · m for feed-forward torque $\Gamma_d(t)$ then control (21) satisfies condition (4) with limit $\Gamma_M = 0.75$ N · m. The chosen feedback gains in the position and velocity of the control law are: $\beta = 0.6$ N · m, $\gamma = 0.8$ N · m/s. Inclination Γ_M/τ has been chosen 1 N · m/s ($\Gamma_M = 0.6$ N · m, $\tau = 0.6$ s).

In Fig. 6-8, the results for $\theta_d = 3$ are displayed. In Fig. 6, feed-forward torque control $\Gamma_d(t)$, reference trajectory in position $\theta_d(t)$ and in velocity $\dot\theta_d(t)$ are shown. In the reference trajectory, time T of the transitional process is 6.16 s. The optimal time for $\Gamma_M = 0.75$ N · m, calculated with formula (13), is 4.94 s. This time is smaller than time 6.16 s by 20%. Torque $\Gamma(t)$, angle $\theta(t)$ and velocity $\dot\theta(t)$ with control law (21) are shown in Fig. 7. Tracking error $\theta_d(t) - \theta(t)$, elastic variable $\delta(t)$ of the joint and its derivative $\dot\delta(t)$ are displayed in Fig. 8. All the variables tend asymptotically to their desired values.

Difference $|\theta_d(t) - \theta(t)|$ is smaller than 0.045 and angle $\theta(t)$ tracks desired angle $\theta_d(t)$ correctly. The behavior of torque $\Gamma(t)$ is similar to that of the elastic variable of joint $\delta(t)$ but in the opposite phase [10]. Its value is always inferior to $\Gamma_M = 0.75$ N $\cdot$ m. After 6.5 s the deviations of all variables from the desired values are small. The deviation of the payload from the desired position is not larger than 7.5 mm. The minimal possible time for the rigid arm (or for rigid variable ϑ) $T = T_s = 4.94$ s is smaller than 6.5 s by 1.56 s or by 23%. Minimal possible time for both variables δ and ϑ is larger than 4.94 s. Consequently, under our control we loose a time smaller than 1.56 s.

Our approach to the control design for a flexible joint arm can be explained also in the following way. We design trapezoidal control, which leads rigid variable ϑ to its desired value ϑ_d. Under this control we deviate from time-optimality for this variable. But due to this fluent control the oscillations of flexible variable δ around its desired motion can be made small. In that case, the final values of variables δ and $\dot{\delta}$ are not far from zero, and the designed control is not far from the time-optimal control.

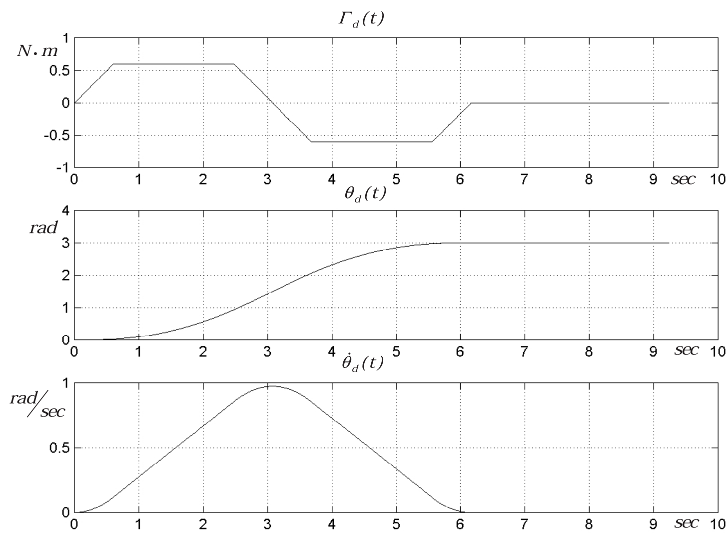

Figure 6. Feed-forward torque and reference trajectory for one-link arm with flexible joint

3. Two-link arm with flexible joints

The two-link manipulator with two elastic joints is studied in this section.

3.1. Mathematical model

The scheme of a two-link arm in a horizontal plane is shown in Fig. 9. The arm is actuated by two motors and can rotate on the plane. The motors are located at motionless

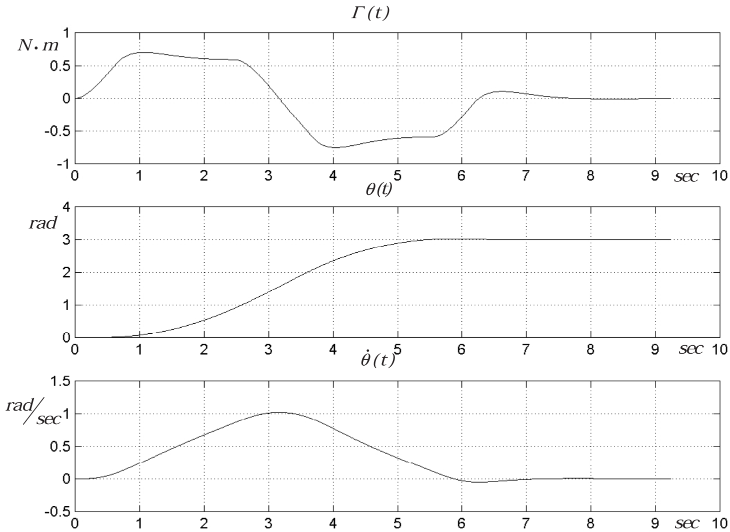

Figure 7. The control torque, angle and angular velocity for one-link arm under control (21)

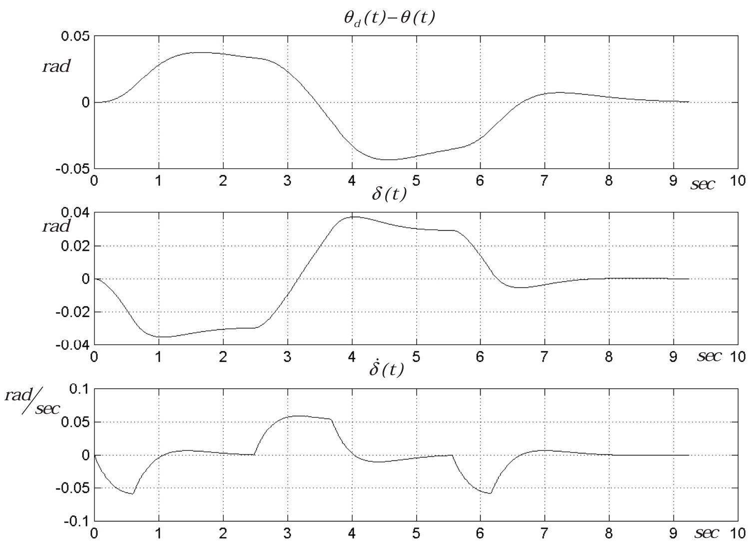

Figure 8. The error tracking of the angle, the elastic variable and its time derivative for one-link arm under control (21)

joint A and at joint B. This arm consists of the two rigid links interconnected by two elastic revolute joints. We assume that the elasticity in the transmission can be modeled as a linear torsional spring located after gear. Using the Lagrange second method, we design the following mathematical model of the two-link manipulator:

$$\begin{pmatrix} a_{11} & a_{12} & 0 & J_{m2} \\ a_{12} & a_{22} & 0 & 0 \\ 0 & 0 & J_{m1} & 0 \\ J_{m2} & 0 & 0 & J_{m2} \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_{m1} \\ \ddot{\theta}_{m2} \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} k_1\left(\theta_1 - \frac{\theta_{m1}}{n_1}\right) \\ k_2\left(\theta_2 - \frac{\theta_{m2}}{n_2}\right) \\ \frac{k_1}{n_1}\left(\frac{\theta_{m1}}{n_1} - \theta_1\right) \\ \frac{k_2}{n_2}\left(\frac{\theta_{m2}}{n_2} - \theta_2\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \Gamma_1 \\ \Gamma_2 \end{pmatrix} \quad (23)$$

with

$$\begin{aligned} a_{11} &= J_1^* + (m_2 + M)L_1^2 + J_2 + 2L_1(ML_2 + m_2 r)\cos\theta_2, \\ a_{12} &= J_2^* + L_1(ML_2 + m_2 r)\cos\theta_2, \\ a_{22} &= J_2^*, \qquad h_1 = -L_1(ML_2 + m_2 r)\sin\theta_2(2\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2, \\ h_2 &= L_1(ML_2 + m_2 r)\sin\theta_2\dot{\theta}_1^2 \end{aligned}$$

Here L_1 and L_2 are the lengths of the first and second link, respectively, r is the distance

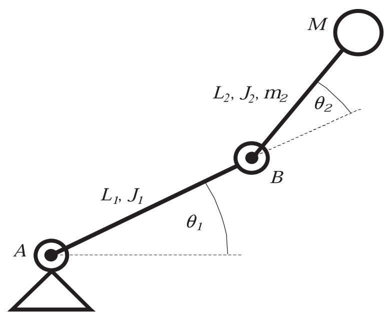

Figure 9. Scheme of an arm with two links

between point B and the mass center of the second link, J_{m1} and J_{m2} are the moments of inertia of each actuator rotor. J_1^* is the sum of the inertia moments about point A of the first link (together with the stator of the motor located at point B) and of the rotor at joint B, as a material point. J_2^* is the sum of the inertia moments about point B of the second link and of the payload. m_2 is the mass of the second link, and M is the mass of the payload. J_2 is the sum of inertia moments about point B of the motor armature at point B, of the second link and of a payload M: $J_2 = J_2^* + J_{m2}$. The stiffness coefficient and gear ratio are denoted by k_i and n_i $(i = 1, 2)$, respectively.

Unlike one-link arm model (1), two-link arm model (23) is nonlinear.

3.2. Statement of the problem

For the two-link arm with two flexible joints, the problem is to find control torques Γ_1, Γ_2 that transfer an arm from its initial state

$$\theta_1(0) = 0, \quad \theta_2(0) = \theta_{20}, \quad \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0 \quad (24)$$

to a final state

$$\theta_1(T) = \theta_{1d}, \quad \theta_2(T) = \theta_{2d}, \quad \dot{\theta}_1(T) = \dot{\theta}_2(T) = 0 \tag{25}$$

and keep it in this state. Here θ_{20} is the initial inter-link angle, θ_{1d} and θ_{2d} are the desired angles of the arm rotation. The initial values of elastic deformations and its derivatives are zero; the final values are desired to be zero too:

$$\delta_1(0)=\delta_2(0)=0, \quad \dot{\delta}_1(0)=\dot{\delta}_2(0)=0, \quad \delta_1(T)=\delta_2(T)=0, \quad \dot{\delta}_1(T)=\dot{\delta}_2(T)=0. \tag{26}$$

Here, $\delta_1 = \theta_1 - \frac{\theta_{m1}}{n_1}$, $\delta_2 = \theta_2 - \frac{\theta_{m2}}{n_2}$. Time T is not given, but it is necessary that this time should be as short as possible.

3.3. Mathematical model of an associated rigid two-link arm

The motion equations of the associated two-link rigid arm (without flexibility in the joints, when $\theta_{mi} = n_i\theta_i$, $i = 1, 2$) are well known [15]:

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} n_1\Gamma_1 \\ n_2\Gamma_2 \end{pmatrix} \tag{27}$$

with

$$b_{11} = J_1 + (m_2 + M)L_1^2 + J_2 + 2L_1(ML_2 + m_2 r)\cos\theta_2,$$

$$b_{12} = b_{21} = n_2 J_{m2} + J_2^* + L_1(ML_2 + m_2 r)\cos\theta_2, \qquad b_{22} = n_2^2 J_{m2} + J_2^*.$$

Here J_1 is the sum of inertia moments about point A of the rotor at joint A (multiplied by n_1^2), of the first link (together with the stator of the motor located at point B) and of the rotor at joint B, as a material point: $J_1 = n_1^2 J_{m1} + J_1^*$. Control torques Γ_1 and Γ_2 of the motors at joints A and B are limited by constants Γ_{1M} and Γ_{2M}:

$$|\Gamma_1| \leqslant \Gamma_{1M}, \qquad |\Gamma_2| \leqslant \Gamma_{2M}. \tag{28}$$

The rigid links of our two-link arm are homogeneous and $r = L_2/2$; the parameters of the arm are the following:

$$L_1 = 1 \text{ m}, \quad L_2 = 0.5 \text{ m}, \quad J_1 = 7.5933 \text{ kg}\cdot\text{m}, \quad J_2 = 0.1534 \text{ kg}\cdot\text{m}^2,$$

$$m_2 = 0.334 \text{ kg}, \quad M = 0.5 \text{ kg},$$

$$J_{m1} = 1.8\cdot 10^{-3} \text{ kg}\cdot\text{m}^2, \quad J_{m2} = 0.185\cdot 10^{-3} \text{ kg}\cdot\text{m}^2, \tag{29}$$

$$\Gamma_{1M} = 0.75 \text{ N}\cdot\text{m}, \quad \Gamma_{2M} = 0.25 \text{ N}\cdot\text{m}, \quad n_1 = n_2 = 5$$

Model (27) of the two-link rigid arm is nonlinear and much more complicated than model (9) of the one-link rigid arm.

3.4. On the time-optimal control of an associated rigid two-link arm

Designing a fast motion for an arm with two flexible joints, we start to design a fast motion for the associated rigid two-link arm. It is not an easy problem to design the time-optimal bounded control for the mechanism with two rigid links and this problem has not been solved in general. We can design the time-optimal control for many cases numerically, using the theory of optimal processes [26]. The structure of this control depends on boundary states (24) - (26) on torque limits Γ_{1M} and Γ_{2M} , and on the arm parameters. There are cases when the time-optimal or quasi-time-optimal motion contains a singular arc [18, 31]. It contains a singular arc, if the robot angle θ_{1d} is 'large', and/or limit value Γ_{1M} is 'small', and/or limit value Γ_{2M} is 'large' [31]. During a singular motion the absolute value of inter-link control torque Γ_2 is smaller than Γ_{2M}, and both links move together. In this case, the two-link rigid arm moves as a one-link arm and its inertia moment about motionless joint A is minimal as possible. Usually, the time-optimal control with limited amplitude has discontinuities. In many situations, it is bang-bang control. In our approach, we suppose that it is possible to design a time-optimal or a quasi-time-optimal torque controls and the corresponding trajectory for an arm with rigid links. Below, we consider the case when the singular motion with coinciding links is possible in a quasi-time-optimal regime. We consider the so-called symmetric movement.

Let us assume that

$$\theta_{2d} + \theta_{20} = -2\pi.$$

Under this assumption, boundary configurations (24) and (25) are symmetric with respect to configuration

$$\theta_1 = \theta_1^* = \theta_{1d}/2, \quad \theta_2 = \theta_2^* = -\pi,$$

because

$$\theta_1(0) = \theta_1^* - \theta_{1d}/2, \qquad \theta_2(0) = \theta_2^* - (\theta_{2d} - \theta_{20})/2,$$
$$\theta_1(T) = \theta_1^* + \theta_{1d}/2, \qquad \theta_2(T) = \theta_2^* + (\theta_{2d} - \theta_{20})/2.$$

System (27) is conservative, if $\Gamma_1 \equiv \Gamma_2 \equiv 0$. Therefore, for an arm with symmetric boundary conditions we search the control torques in symmetric form [16]:

$$\Gamma_1(t) + \Gamma_1(T-t) \equiv 0, \qquad \Gamma_2(t) + \Gamma_2(T-t) \equiv 0, \qquad (0 \leqslant t \leqslant T). \tag{30}$$

It is easy to prove analytically that the solution of system (27) under symmetric control torques is symmetric too (the numerical investigations confirm this symmetry of course):

$$\begin{aligned} &\theta_1(t) + \theta_1(T-t) \equiv \theta_{1d}, \quad \theta_2(t) + \theta_2(T-t) \equiv -2\pi, \\ &\dot\theta_i(t) - \dot\theta_i(T-t) \equiv 0, \qquad \ddot\theta_i(t) + \ddot\theta_i(T-t) \equiv 0 \qquad (i = 1, 2). \end{aligned} \tag{31}$$

If $-\pi < \theta_{20} \leqslant 0$, angle θ_{1d} is 'large', and/or value Γ_{1M} is 'small', and/or value Γ_{2M} is

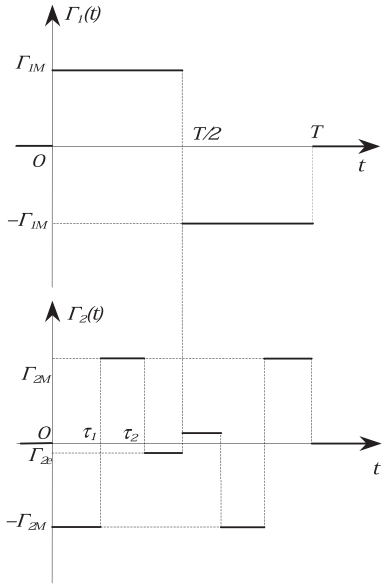

Figure 10. Discontinuous quasi-time-optimal control functions

'large', then we can search control functions $\Gamma_1(t)$, $\Gamma_2(t)$ for a two-link rigid arm with the singular arc in the following form [18] (see Fig. 10):

$$\Gamma_1(t) = \begin{cases} \Gamma_{1M}, & \text{if} \quad 0 \leqslant t < T/2, \\ -\Gamma_{1M}, & \text{if} \quad T/2 \leqslant t \leqslant T, \\ 0, & \text{if} \quad t < 0 \text{ or } t > T. \end{cases} \tag{32}$$

$$\Gamma_2(t) = \begin{cases} -\Gamma_{2M}, & \text{if} \quad 0 \leqslant t < \tau_1, \\ \Gamma_{2M}, & \text{if} \quad \tau_1 \leqslant t < \tau_2, \\ \Gamma_{2e}, & \text{if} \quad \tau_2 \leqslant t < T/2, \\ -\Gamma_{2e}, & \text{if} \quad T/2 \leqslant t < T - \tau_2, \\ -\Gamma_{2M}, & \text{if} \quad T - \tau_2 \leqslant t < T - \tau_1, \\ \Gamma_{2M}, & \text{if} \quad T - \tau_1 \leqslant t \leqslant T, \\ 0, & \text{if} \quad t < 0 \text{ or } t > T. \end{cases} \tag{33}$$

Here

$$\Gamma_{2e} = b_{12}\frac{\Gamma_{1M}}{b_{11}} \tag{34}$$

We have to calculate values b_{11}, b_{12} in formula (34) for $\theta_2 = -\pi$. Consequently, in the time interval

$$\tau_2 \leqslant t < T - \tau_2 \tag{35}$$

there is the identity

$$\Gamma_2(t) \equiv b_{12}\frac{\Gamma_1(t)}{b_{11}} \tag{36}$$

Discontinuous control functions (32) and (33) satisfy symmetric conditions (30). Control torque (36) has to satisfy the second inequality (28). If it satisfies this condition strictly, then control (33) is singular [18] in interval (35).

Studying system (27), (32) – (34) numerically, we first try to find switching times τ_1, τ_2 so that

$$\theta_2(\tau_2) = -\pi, \qquad \dot{\theta}_2(\tau_2) = 0. \tag{37}$$

In interval (35), under conditions (37) and control (36) and (34), we have the identity

$$\theta_2(t) \equiv -\pi. \tag{38}$$

Thus, in time-interval (35), where the motion is singular, both links move together. It means the two-link rigid arm moves as a one-link arm and its inertia moment with respect to point A is minimal, as possible.

Equality $\theta_1(T/2) = \theta_{1d}/2$ follows from the first identity (31). After detecting instants τ_1 and τ_2, numerically we find time T so that $\theta_1(T/2) = \theta_{1d}/2$. We have numerically investigated system (27), (32) – (34) with parameters (29) and the following configurations (see Fig. 11):

$$\theta_1(0) = \theta_2(0) = \theta_{20} = 0, \quad \theta_1(T) = \theta_{1d} = 3, \quad \theta_2(T) = \theta_{2d} = -2\pi. \tag{39}$$

Calculated torques $\Gamma_i(t)$, angles $\theta_i(t)$, angular velocities $\dot{\theta}_i(t)$ and accelerations $\ddot{\theta}_i(t)$, $(i = 1, 2)$ satisfy symmetry conditions (30) and (31). Functions $\ddot{\theta}_i(t)$, $(i = 1, 2)$ have discontinuities and functions $\dot{\theta}_i(t)$, $(i = 1, 2)$ are not smooth. The process time is $T = 4.98$ s. The designed control is close to the time-optimal control with restrictions (28) [16, 18].

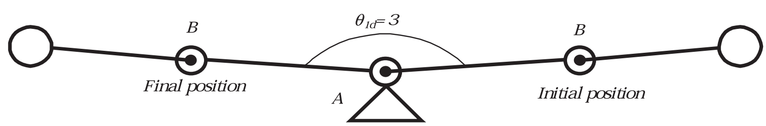

Figure 11. Boundary configurations of an arm with two links

3.5. On the design of fluent control and corresponding trajectory

It is shown theoretically and experimentally (see: [4, 5] and section 2 in this paper), that the discontinuous control of kind (32) and (33) is not acceptable for an arm with flexible links or flexible joints because large elastic oscillations appear. We employ here the so-called 'fluent' control [2]. Functions (32) and (33) are changed to continuous functions $\Gamma_{1d}(t)$ and $\Gamma_{2d}(t)$ of the following 'trapezoidal' form [4, 5] (see Fig. 12):

$$\Gamma_{1d}(t) = \begin{cases} \frac{\Gamma_{1M}}{\tau}, & \text{if} \quad 0 \leqslant t < \tau, \\ \Gamma_{1M}, & \text{if} \quad \tau \leqslant t < T/2 - \tau, \\ \frac{\Gamma_{1M}}{\tau}(T/2 - \tau), & \text{if} \quad T/2 - \tau \leqslant t < T/2, \\ -\Gamma_{1d}(T/2 - t), & \text{if} \quad T/2 \leqslant t \leqslant T, \\ 0, & \text{if} \quad t < 0 \text{ or } t > T. \end{cases} \tag{40}$$

$$\Gamma_{2d}(t) = \begin{cases} -\frac{\Gamma_{2M}}{\sigma}t, & \text{if} \quad 0 \leqslant t < \sigma, \\ -\Gamma_{2M}, & \text{if} \quad \sigma \leqslant t < t_1 - \sigma, \\ -\Gamma_{2M} + \frac{\Gamma_{2M}}{\sigma}(t - t_1 + \sigma), & \text{if} \quad t_1 - \sigma \leqslant t < t_1 + \sigma, \\ \Gamma_{2M}, & \text{if} \quad t_1 + \sigma \leqslant t < t_2 - \sigma, \\ \Gamma_{2M} - \frac{\Gamma_{2M}}{\sigma}(t - t_2 + \sigma), & \text{if} \quad t_2 - \sigma \leqslant t < t_2 - \sigma\frac{\Gamma_{2e}}{\Gamma_{2M}}, \\ \Gamma_{2e}, & \text{if} \quad t_2 - \sigma\frac{\Gamma_{2e}}{\Gamma_{2M}} \leqslant t < T/2 - \tau, \\ \Gamma_{2e} - \frac{\Gamma_{2e}}{\tau}(t - \frac{T}{2} + \tau), & \text{if} \quad T/2 - \tau \leqslant t \leqslant T/2, \\ -\Gamma_{2d}(T - \tau), & \text{if} \quad T/2 \leqslant t \leqslant T, \\ 0, & \text{if} \quad t < 0 \text{ or } t > T. \end{cases} \tag{41}$$

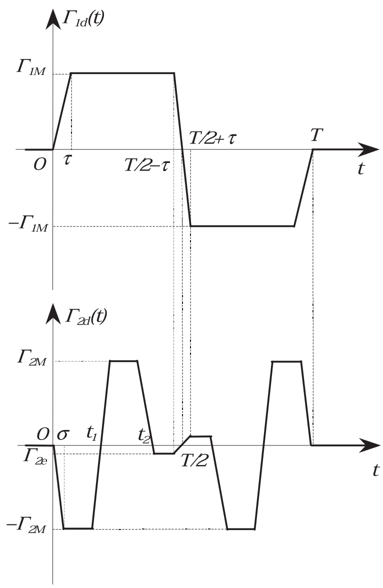

Figure 12. Fluent control functions

Continuous control functions (40) and (41) tend to discontinuous functions (32) and (33) respectively, if τ and σ tend to zero. Both functions (40), (41) are symmetric in interval $[0, T]$ with respect to point $(T/2, 0)$: $\Gamma_{id}(t) \equiv -\Gamma_{id}(T - t)$, $(i = 1, 2)$. For given instants τ and σ we find switching times t_1, t_2 and final time T by numerical integrating system (27) under control (40), (41) and (34). To find these values the process of iterations was organized. Switching times τ_1, τ_2 and final time T from discontinuous control (32) and (33) could be used as a first approximation in this process of iterations. At first, we find instants t_1 and t_2 so that

$$\theta_2(t_2 - \sigma\Gamma_{2e}/\Gamma_{2M}) = -\pi, \quad \dot{\theta}_2(t_2 - \sigma\Gamma_{2e}/\Gamma_{2M}) = 0. \tag{42}$$

Under conditions (42) and control (41) in the time-interval

$$t_2 - \sigma\Gamma_{2e}/\Gamma_{2M} \leqslant t \leqslant T - t_2 + \sigma\Gamma_{2e}/\Gamma_{2M},$$

identity (38) exits and the two-link rigid arm moves as a one-link arm with minimal inertia moment around motionless joint A. After detecting instants t_1 and t_2, we numerically search time T so that $\theta_1(T/2) = \theta_{1d}/2$.

We deviate from the time-optimality, of course, using for the two-link rigid arm fluent control (40) and (41) instead of discontinuous input (32) and (33). But the deviation is small, if ratios Γ_{1M}/τ and Γ_{2M}/σ are large (times τ and σ are small).

Control functions (40), (41) for system (27) with parameters (29) and symmetric boundary configurations (39) are designed numerically (see Figs 13 and 16). Torques $\Gamma_{id}(t)$, corresponding angles $\theta_{id}(t)$, angular velocities $\dot{\theta}_{id}(t)$ and accelerations $\ddot{\theta}_{id}(t)$, $(i = 1, 2)$ which were found for $\tau = 0.75$ s, $\sigma = 0.25$ s satisfy symmetry conditions (30) and (31) in interval $0 \leqslant t \leqslant T$. Unlike the case of discontinuous control, functions $\ddot{\theta}_{id}(t)$, $(i = 1, 2)$ are continuous and functions $\dot{\theta}_{id}(t)$, $(i = 1, 2)$ are smooth. Final time is $T = 5.78$ s. Final time 4.98 s for discontinuous control (32), (33) is smaller than 5.78 s by 14% approximately.

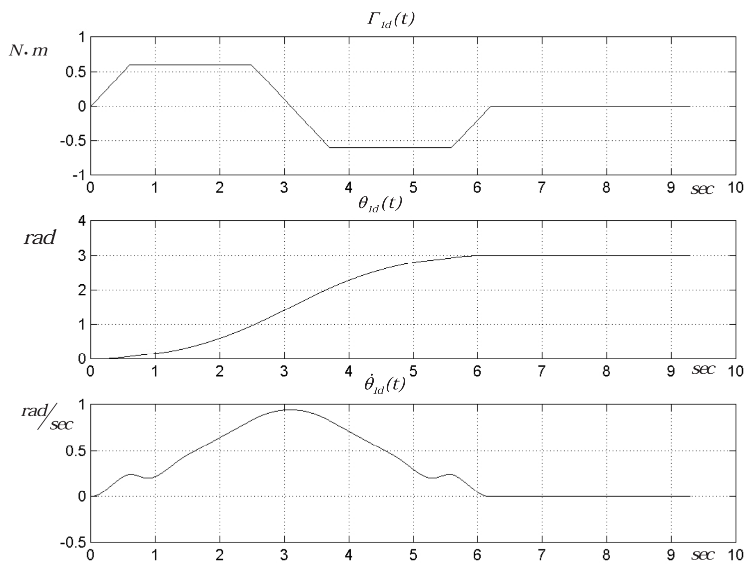

Figure 13. Feed-forward torque and reference trajectory for the first link of two-link arm with flexible joints

3.6. Numerical experiment with flexible two-link arm

We choose torque control Γ_i, $(i = 1, 2)$ that consists of commanded feed-forward torque $\Gamma_{id}(t)$ combined with a linear position and velocity feedback [4, 5, 8, 30]

$$\Gamma_i = \Gamma_{id}(t) + \beta_i[n_i\theta_{id}(t) - \theta_{mi}] + \gamma_i[n_i\dot{\theta}_{id}(t) - \dot{\theta}_{mi}] \quad (i = 1, 2). \qquad (43)$$

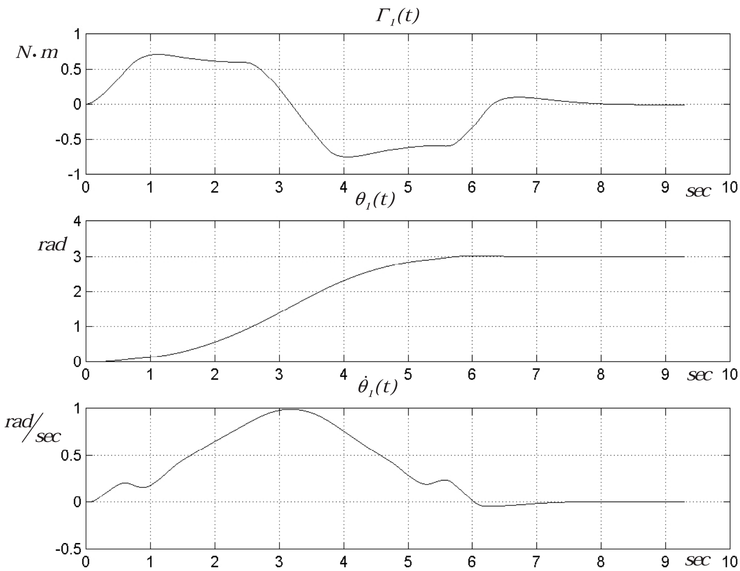

Figure 14. The control torque, angle and angular velocity for the first link of two-link arm under control (43)

Thus, we use calculated torques $\Gamma_{id}(t)$ and functions $\theta_{id}(t)$, $\dot{\theta}_{id}(t)$, $(i = 1, 2)$ as input signals for usual PD-controllers. Note that information about deformation in the flexible joints is not used in the proposed control. For the numerical experiments the parameters of the arm with homogeneous link are given by (29). The stiffness coefficients are the following: $k_1 = k_2 = 100$ N·m. Calculating feed-forward torques $\Gamma_{id}(t)$, $(i = 1, 2)$ we have decreased slightly limits Γ_{1M} and Γ_{2M} like for the one-link arm case because in control (43), as well as feed-forward torque $\Gamma_{id}(t)$, there are terms with feedback signals too. Limits $\Gamma_{1M} = 0.6$ N·m and $\Gamma_{2M} = 0.2$ N·m have been chosen for feed-forward torques. The chosen feedback gains in position and velocity of the control law are: $\beta_i = 0.6$ N·m, $\gamma_i = 0.8$ N·m/s, $(i = 1, 2)$.

The computer calculates before an experiment continuous control functions $\Gamma_{1d}(t)$, $\Gamma_{2d}(t)$ in form (40) and (41) as it is written in the previous subsection. During the motion of the two-link arm with flexible joints, the computer solves on-line motion equations (27) of the rigid two-link arm with $\Gamma_1 = \Gamma_{1d}(t)$, $\Gamma_2 = \Gamma_{2d}(t)$ and calculates the corresponding behavior of angles $\theta_{id}(t)$ and angular velocities $\dot{\theta}_{id}(t)$, $(i = 1, 2)$. This behavior of the associated rigid two-link arm is used as the reference trajectory.

In Figs 13-15 and 16-18, the results for $\theta_{1d} = 3$ and $\theta_{2d} = -2\pi$ are displayed. In Figs 13 and 16, feed-forward control torques $\Gamma_{id}(t)$, reference trajectories in position $\theta_{id}(t)$ and in velocity $\dot{\theta}_{id}(t)$, $(i = 1, 2)$ are shown. Torques $\Gamma_i(t)$, angles $\theta_i(t)$ and velocities $\dot{\theta}_i(t)$, $(i = 1, 2)$ with control (43) are shown in Figs 14 and 17. Tracking errors

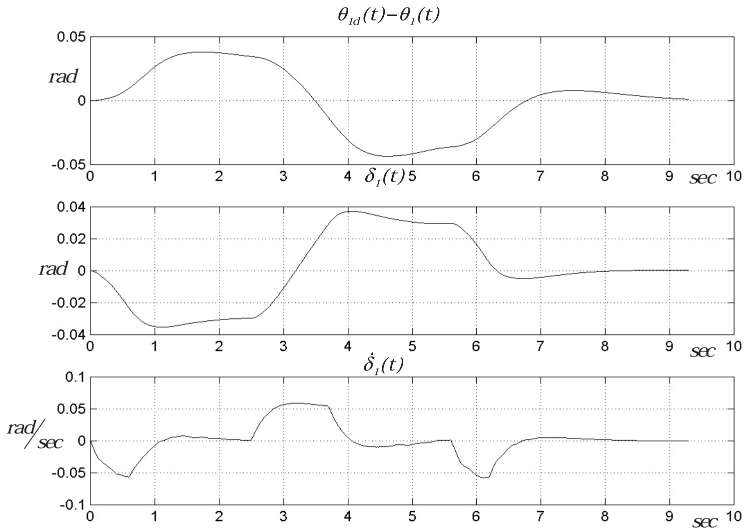

Figure 15. The error tracking of the angle, the elastic variable and its time derivative for the first link of two-link arm under control (43)

$\theta_{id}(t) - \theta_i(t)$, joint elastic variables $\delta_i(t)$ and its derivatives $\dot{\delta}_i(t)$, $(i = 1, 2)$ are displayed in Figs 15 and 18. All the variables tend asymptotically to their desired values. Differences $|\theta_{id}(t) - \theta_i(t)|$, $(i = 1, 2)$ are smaller respectively than 0.045 and 0.012 during all time. Each angle $\theta_i(t)$ tracks desired angle $\theta_{id}(t)$ correctly $(i = 1, 2)$. The maximal value of deformation $|\delta_2(t)|$ for the second link is near 0.01, at the same time the amplitude of the oscillations around 'stationary process' of the deformation changes is smaller than 0.002. The behavior of torques $\Gamma_i(t)$, $(i = 1, 2)$ is similar to the ones of elastic joint variable $\delta_i(t)$, but in the opposite phase [10]. Their values are always inferior to $\Gamma_{1M}(t) = 0.75$ N·m and $\Gamma_{2M}(t) = 0.25$ N·m. After 6.6 s, the deviations of all variables from the desired values are small. The deviation of the payload from the desired position is smaller than 5.5 mm. If, for a concrete technological operation, such deviations are acceptable, we say the transition process has finished to time 6.6 s (time 4.98 s for discontinuous control (32), (33) is smaller than 6.6 s by 25%). Otherwise, we have to wait until the deviations become admissible or to decrease ratios Γ_{1M}/τ and/or Γ_{2M}/σ. In both cases, the duration of transition process increases.

4. Conclusion

In this paper, the analytical and numerical study of the control problem of one-link arm with one flexible joint and two-link anthropomorphic arm with two flexible joints is presented. We proposed a method to design both feed-forward control torques and

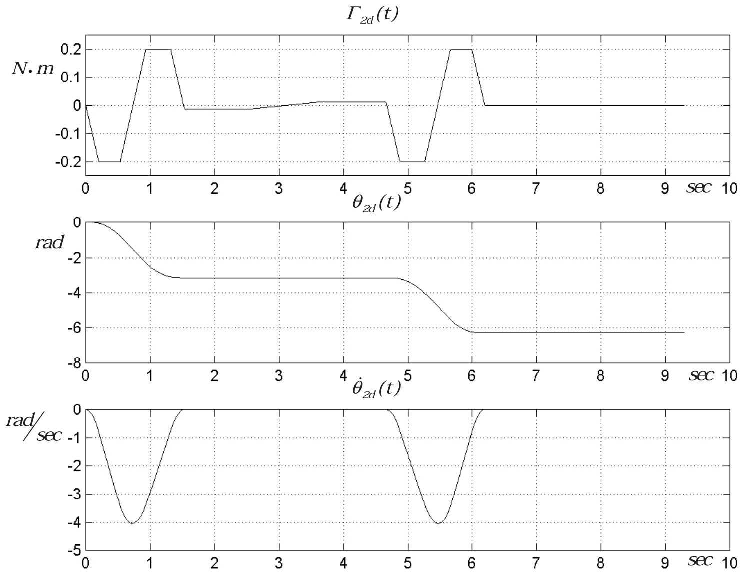

Figure 16. Feed-forward torque and reference trajectory for the second link of two-link arm

reference arm trajectory. The limits on the torques are taken into account explicitly and designed feed-forward torques are quasi-time-optimal one. The designed feed-forward torques and reference trajectory are used as input signals for usual PD-controllers.

A high-quality controlled system has to be fast. But the mathematical model of an arm with two flexible joints is highly nonlinear and complex. Therefore, it is very difficult to design the time-optimal bounded control for this object. Besides what it is very important that the stiffness coefficients are not well known. It means that model of an arm with flexible joints is not well known. At the same time, the model of an arm without elastic joints is simpler, well known and it is easier to design the time-optimal or the quasi-time-optimal control for it. For these reasons, we adjust a time-suboptimal control for a rigid arm (without elastic joints) to a flexible arm (with elastic joints). At first, we compute for the associated two-link rigid arm, the control which is quasi-time-optimal. This control contains discontinuities. This is standard in the case of a bounded control. If we employ this discontinuous control for an arm with flexible joints, large elastic oscillations appear in the joints, because the switching times computed for the associated rigid arm are not appropriate for the flexible arm. The damping of the resulting vibrations requires a long time in this case. To avoid large elastic vibrations, we modify the obtained discontinuous torque controls by continuous functions that are close to the discontinuous functions. The modification of discontinuous control functions depends on the elasticity of the arm joints. We prove (analytically) for an individual link with a flexible joint the possibility to make the elastic vibrations as small as desired. Under modified

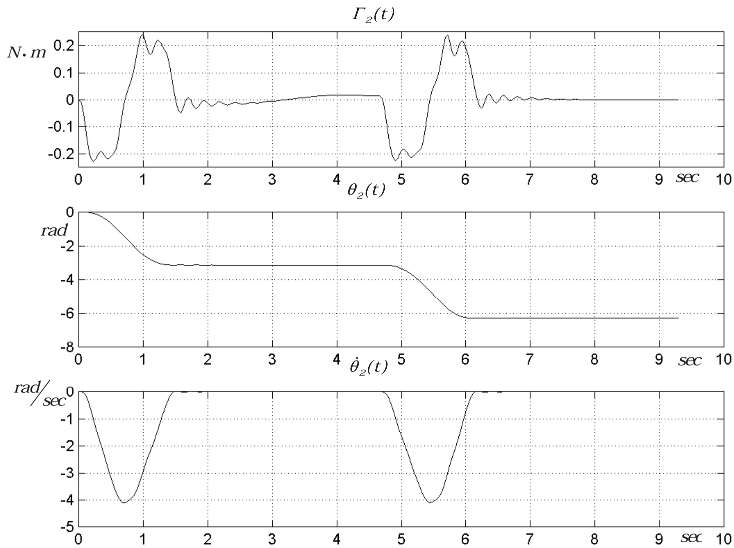

Figure 17. The control torque, angle and angular velocity for the second link of two-link arm under control (43)

continuous control the amplitude of the elastic vibrations is limited and the behaviors of rigid variables θ_1, θ_2 for flexible and rigid arms are close. It confirms the applicability of our control design method for a flexible joint arm, using a model of a rigid arm. In the numerical experiments with the flexible joint arm, we successfully use these continuous control functions as commanded feed-forward torques, the corresponding angles and the angular velocities as a reference trajectory for the position and the velocity feedback. The motion of the one-link arm with flexible joint or the two-link arm with two flexible joints is not far from the time-optimal motion.

Some authors try to design a control, which quickly damps the elastic vibrations excited in the flexible joints. But this problem of damping is not easy. Therefore, in our opinion, the problem of control design for an arm with flexible joints is not closed. We propose here another approach. A control, which does not excite unwanted large elastic vibrations, is designed. The important point, using this approach is that no information on line about the deformation in the elastic joints is necessary. This developed method might be of some interest to the control theory for a manipulator with flexible joints, and industrial applications.

It seems to us that our approach can be recommended not only for a flexible manipulator, but also for other systems with a flexible structure and for some under-actuated systems.

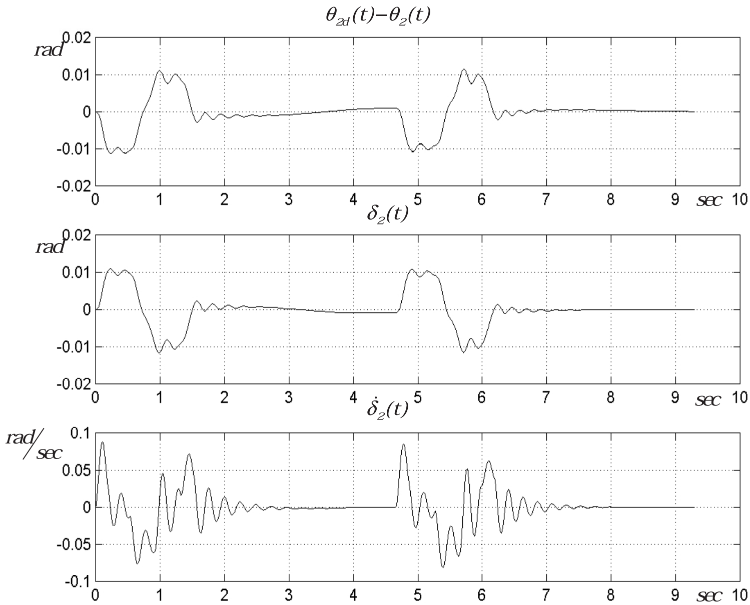

Figure 18. The error tracking of the angle, the elastic variable and its time derivative for the second link of two-link arm under control (43)

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