*Archives of Control Sciences*
*Volume* 12(XLVIII), 2002
*No.* 1–2*, pages 7–36*

# Feedback control of nonholonomic wheeled vehicles. A survey.

PASCAL MORIN and CLAUDE SAMSON

The paper is an introduction and overview to the problem of feedback control of nonholonomic wheeled vehicles. Solutions proposed during the last decade and a new approach currently developed by the authors of this article are presented.

**Key words:** wheeled mobile robots, nonholonomic systems, driftless nonlinear systems, feedback control, stabilization

## 1. Introduction

The purpose of this article is to present the main issues associated with the problem of state feedback control of wheeled vehicles subjected to nonholonomic constraints. It is also to provide researchers in Robotics who are not specialists in Control Theory with a few keys to decode the existing literature on the subject. This is not a collection of ready-to-use control schemes, but rather a reflexion on some developments made during the last decade. Also, in the last part of the article, a new approach to *practical* stabilization based on the use of *transverse functions* is presented. Our desire to produce a didactic document with a touch of generality and depth at the conceptual level led us to immerse the problem into the more general one of control of driftless nonlinear systems. The proposed synthesis is limited to feedback regulation aspects. Bibliographical landmarks are nevertheless given in the last section of the article, some of them pointing at important issues and extensions of the control problem which are not addressed here: open loop control, explicit calculation of trajectories/solutions, processing of information data acquired by sensors, either proprioceptive (odometry) or exteroceptive (camera, sonar, laser rangefinder,...), for state reconstruction and use in the control loops (sensor-based control).

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The Authors are with the INRIA, 2004, Route des Lucioles, 06902 Sophia-Antipolis Cedex, France.

A French version of this article is in the proceedings of the *Troisièmes journées Nationales de la Recherche en Robotique* (JNRR'01), Hyères, October 15-17, 2001, INRIA Ed.

## 2. Modeling aspects

It is well known that the kinematics of nonholonomic wheeled vehicles can be modelled, under the classical assumption of *rolling without slippage*, by equations in the form

$$\dot{x} \;=\; \sum_{i=1}^{m} b_i(x)u_i \qquad b_i \in C^{\infty}(\Omega \subset \mathbb{R}^n;\mathbb{R}^n)\ ,\forall i \in \{1,\ldots,m\} \tag{1}$$

with $x$ a configuration vector (position + orientation) – or *attitude* – of the system, and the $u_i$'s standing for independent velocity variables whose number is equal, by definition, to the number of degrees of freedom of the mechanical system.

For example, a kinematic model of a *unicycle-type* vehicle – with two independent tracting wheels on a common axle –, schematized in the view-from-above of Figure 1, is

$$\begin{aligned} \dot{x}_1 &= u_1 \cos x_3 \\ \dot{x}_2 &= u_1 \sin x_3 \\ \dot{x}_3 &= u_2 \end{aligned} \tag{2}$$

with $u_1$ and $u_2$ the respective (signed) intensities of the point $P$ velocity vector and vehicle's angular velocity.

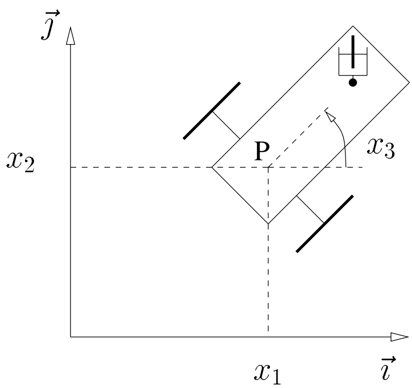


Figure 1. Unicyle-type vehicle

In the case of a tricycle, a *car-like* vehicle schematized in the view-from-above of Figure 2, the corresponding kinematic model is

$$\begin{aligned} \dot{x}_1 &= u_1 \cos x_3 \\ \dot{x}_2 &= u_1 \sin x_3 \\ \dot{x}_3 &= \frac{u_1}{L} \tan x_4 \\ \dot{x}_4 &= u_2 \end{aligned} \tag{3}$$

with $u_2$ denoting now the intensity of the angular velocity of the front driving wheel about the axis perpendicular to the plane of motion of the vehicle, and $L$ the distance between front and rear wheels' axles.

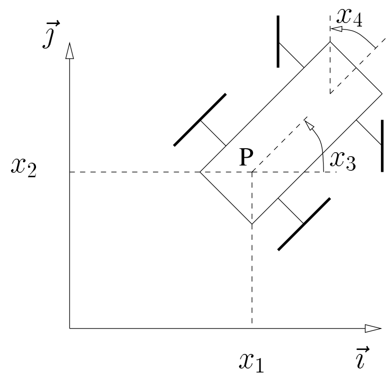


Figure 2. Tricycle or car-like vehicle

In the case of the unicycle we thus have $n = 3$ and $m = 2$, while in the case of the tricycle $n = 4$ and $m = 2$. Hooking trailers to these vehicles increases the dimension $n$ of the configuration vector (by one per each trailer), but does not modify the number of degrees of freedom (which thus remains equal to two). The possibility of interpreting the velocity variables $u_1$ et $u_2$ as *control variables* results from the possibility of associating each degree of freedom with physical means of actuation and using the corresponding actuator to produce a *desired velocity* value. Of course, any attempt to model the system and its actuators more completely would yield a larger state vector and control inputs which, depending on the chosen level of modeling refinement, would be either accelerations of joint angles, or torques produced by the actuators in order to obtain these accelerations, or current intensities through the actuators to produce the required torques (in the case of conventional D.C. motors), or numerical inputs to AC/DC power converters in order to produce the desired currents,... We will only consider here basic kinematic models which already account for the difficulties associated with the system's nonholonomy. Considering more complete, or refined, models would render the exposition more technical without providing a better conceptual insight of the problems.

## 3. Controllability aspects

Kinematic models of nonholonomic vehicles present a few specific features. The first one is that the dimension $n$ of the state vector is always strictly larger than the number $m$ of (independent) control variables. From the preceding discussion, the difference between these two numbers can only increase with the degree of refinement brought to the model. Recall that these numbers are equal in the case of holonomic mechanisms – manipulator arms, for instance – for which a kinematic model, $\dot{x} = u$, merely indicates that each configuration variable is associated with a degree of freedom. The difference between $n$ and $m$ is also – under the assumption of *full actuation*, made here, according to which each degree of freedom is paired with its own physical actuating device – the main reason for the difficulty to control nonholonomic systems.

The second important characteristic of these systems, with respect to the control issue, is that the **linear approximation of such a system at any equilibrium point** $(x = x^\star, u = 0)$ **is not controllable**, nor even stabilizable. Indeed, the equation of this approximation is

$$\dot{x} = \sum_{i=1}^{m} b_i(x^\star)u_i \tag{4}$$

and the associated controllability matrix, which in the present case is just the matrix made of the $m$ column vectors $b_i(x^\star)$, has its rank at most equal to $m$ (thus less than $n$). It is simple to verify on the examples (2) and (3) that the rank of this matrix reaches in fact its maximal value $m$, whatever the configuration $x^\star$. Since this rank property can be extended to other nonholonomic vehicles, we will assume from now on that $\{b_1(x), \ldots, b_m(x)\}$ is a set of independent vectors in $\mathbb{R}^n$, for any $x$. The non-stabilizability of the linear approximation of the system implies by itself that *no nonlinear system* (1) *can be linearized by feedback* (either static or dynamic) in the neighborhood of a fixed configuration. Therefore, classical nonlinear control techniques based on linearization, either approximated or exact, of the original system cannot provide a solution to the problem of *feedback asymptotic stabilization* of a fixed configuration[1].

The defect of controllability of the linearized system (4) may suggest that the original nonlinear system is not controllable either. But this cannot be the case for fully actuated nonholonomic mechanical systems. First of all, *common sense* tells us that wheeled vehicles (cars, in particular) would not be produced in large quantities if they were not controllable. More rigorously, one can show by application of Frobenius Theorem – and by definition of nonholonomic constraints (non-completely integrable bilateral constraints depending on variable configurations and their time derivatives) – that there always exist systems of coordinates (sets of configuration variables) for which some of the coordinates are controllable while the others are constant – e.g. cannot be modified by any control action. It is then natural to keep only the former ones in a kinematic model of the system. Note that the same remark applies to holonomic systems. For instance, the (non-controllable) coordinates of the fixed base of a manipulator arm never appear in the modeling equations of the system. Similarly, in the case of a train – an example of a “holonomic” wheeled vehicle – it is only necessary to know the curvilinear coordinate of one point on the train along the rail track to be able to determine the train’s configuration. Besides, the controllability of systems (2) and (3) is easily checked analytically by application of Chow’s Theorem (1939) which says that a system in the form (1) is locally controllable at a point $x^\star$ if and only if the following rank condition, called *Lie*

[1]We recall that the property of asymptotic stability of an equilibrium point of a system $\dot{x} = f(x,t)$ is the conjunction of the property of stability of this point in the sense of Lyapounov and the convergence to this point of all solutions initiated in a neighborhood.

*Algebra Rank Condition (LARC)*, is satisfied. [2]

$$dim\ (Span\{b(x^\star) : b \in Lie(b_1, \ldots, b_m)\}) \ = \ n \tag{5}$$

For instance, in the case of the unicycle (2) this condition is satisfied at any point because the vectors

$$b_1(x) = \begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[b_2, b_1](x) \triangleq \frac{\partial b_1}{\partial x}(x) b_2(x) - \frac{\partial b_2}{\partial x}(x) b_1(x) = \begin{pmatrix} -\sin x_3 \\ \cos x_3 \\ 0 \end{pmatrix}$$

are clearly linearly independent at any point $x$. This condition is also satisfied for the tricycle because the vectors

$$b_1(x) = \begin{pmatrix} \cos x_3 \\ \sin x_3 \\ \frac{\tan x_4}{L} \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[b_2, b_1](x) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{L \cos^2 x_4} \\ 0 \end{pmatrix}, \quad [[b_2, b_1], b_1](x) = \begin{pmatrix} -\frac{\sin x_3}{L \cos^2 x_4} \\ \frac{\cos x_3}{L \cos^2 x_4} \\ 0 \\ 0 \end{pmatrix}$$

are linearly independent for every $x$ such that $x_4 \neq \frac{\pi}{2} + k\pi$ ($k \in \mathbb{Z}$). At singular configurations for which this latter condition is not satisfied, i.e. for which the driving wheel is perpendicular to the vehicle's longitudinal axis, the system remains controllable. Indeed, the apparent problem comes from the non-physical choice, implicitly made in the model (3), of considering the intensity of the vehicle's longitudinal velocity as a free control variable. Now, it is clear that this intensity has to be equal to zero when the angle of the driving wheel is perpendicular to the vehicle's longitudinal axis. To remove this *singularity of representation* it suffices, for instance, to set $u_1 = v_1 \cos x_4$ and take $v_1$ as the control variable.

Since the mechanical systems considered here are *locally controllable at any point* in the configuration space, we can thus make the assumption that the vector fields $b_1, \ldots, b_m$ of the general model (1) satisfy the LARC condition (5) for every $x$ belonging to the model's domain of definition $\Omega \subset \mathbb{R}^n$.

[2] $Lie(b_1, \ldots, b_m)$ denotes the *Lie Algebra* generated by the *vector fields* $b_1, \ldots, b_m$.

## 4. Modelling by homogeneous systems. The particular case of chained systems

In order to simplify the forthcoming discussion, without unduly restricting the generality of the analysis, we make an additional assumption: we will assume from now on that the vector fields $b_1, \ldots, b_m$ are *homogeneous of the same degree with respect to some dilation*. Basic definitions associated to the concept of homogeneity are recalled in the appendix – see [16] for more details on the subject. This assumption of homogeneity reduces the set of studied systems by inducing structural properties which may in turn be exploited at the control design level. It implies for instance – by the assumption of differentiability at any order made upon the vector fields $b_i$ – that the components of the functions $x \mapsto b_i(x)$ are polynomial. It is not very restrictive for two reasons. The first one is that the kinematic equations of some mechanical systems, like unicycle-type and tricycle-like vehicles, can be *semi-globally* modeled by homogeneous systems. By "semi-globally" we mean that limitations upon the domain of validity of the model do not arise from the truncation of functions' Taylor's expansions – whose validity is *local* by nature –, as in the case of the linear approximation of a system about an equilibrium point, but from singularities of representation associated with a change of coordinates whose domain of definition is strictly smaller than $\mathbb{R}^n$. For example, by making the change of coordinates $x \longmapsto \bar{x}$, defined on $\mathbb{R} \times \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ by

$$\bar{x}_1 \triangleq x_1,\ \bar{x}_2 \triangleq \tan x_3,\ \bar{x}_3 \triangleq x_2,$$

and by setting

$$\bar{u}_1 \triangleq u_1 \cos x_3,\ \bar{u}_2 \triangleq \frac{u_2}{\cos^2 x_3}$$

the model of the unicycle (2) can be equivalently written as

$$\begin{aligned} \dot{\bar{x}}_1 &= \bar{u}_1 \\ \dot{\bar{x}}_2 &= \bar{u}_2 \\ \dot{\bar{x}}_3 &= \bar{u}_1 \bar{x}_2 \end{aligned} \tag{6}$$

In the case of a car-like vehicle, the following change of coordinates, defined on the set $\mathbb{R} \times \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ by

$$\bar{x}_1 \triangleq x_1,\ \bar{x}_2 \triangleq \frac{\tan x_4}{L \cos^3 x_3},\ \bar{x}_3 \triangleq \tan x_3,\ \bar{x}_4 = x_2,$$

combined with the introduction of the new control variables

$$\bar{u}_1 \triangleq u_1 \cos x_3,\ \bar{u}_2 \triangleq \frac{1}{L \cos^3 x_3 \cos^2 x_4} \left( u_2 + 3 \frac{u_1}{L} \tan x_3 \sin^2 x_4 \right)$$

transforms the model (3) into

$$\begin{aligned} \dot{\bar{x}}_1 &= \bar{u}_1 \\ \dot{\bar{x}}_2 &= \bar{u}_2 \\ \dot{\bar{x}}_3 &= \bar{u}_1 \bar{x}_2 \\ \dot{\bar{x}}_4 &= \bar{u}_1 \bar{x}_3 \end{aligned} \tag{7}$$

The systems (6) and (7) are the first two elements of a class of homogeneous systems called *chained systems*, and more specifically of the subclass of *monochained systems with two inputs* whose general equation is

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1 x_2 \\ &\vdots \\ \dot{x}_n &= u_1 x_{n-1} \end{aligned} \tag{8}$$

The two vector fields of this chained system are

$$b_1(x) \triangleq \begin{bmatrix} 1 \\ 0 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} \qquad b_2 \triangleq \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

One easily verifies, from the definitions recalled in the appendix, that the control fields associated with the system (8) are homogeneous of degree $-1$ with respect to the dilation defined by

$$\Delta_\lambda^r x \triangleq (\lambda x_1, \lambda x_2, \lambda^2 x_3, \ldots, \lambda^{n-1} x_n)$$

**Remark :** Chronologically, chained systems have been introduced in studies about modeling and control of mobile robots before the more general class of controllable homogeneous driftless systems. They have, among these latter systems, a very specific structure which appears not only in the writing of the equations but also, more significantly, in the fact that their dimension coincides with the dimension of the *Lie Algebra* generated by the two control vector fields (whereas the former dimension is, in the general case, a lowerbound of the second one). It is thus remarkable, and also a little mysterious (because this is probably not a coincidence), that these systems can represent a fair number of human-built physical systems, among which the wheeled vehicles which are of particular interest to us here. Nonetheless, the representativity of these systems should not be overestimated either since many other physical systems that can be modeled by a driftless system like (1) cannot be put in the chained form. For instance, it is well known that, apart a very specific case, a vehicle pulling more than one trailer cannot be modeled or approximated, in a large operational domain, by a chained system [42].

Let us now focus our attention on the notion of *homogeneous approximation* of a driftless system (1). The second reason why the assumption of homogeneity is not very rectrictive comes from that a controllable system (1), even if it is not homogeneous, can always be approximated —locally, around an equilibrium point— by an homogeneous driftless system which is also controllable [16, 49] and whose vector fields share the same degree of homogeneity (that may be set equal to $-1$) with respect to some dilation. Note that the classical linear approximation of a system is a particular case of homogeneous approximation with respect to the dilation defined by $\Delta_\lambda^r x = \lambda x$. A shortcoming of the linear approximation —perfectly illustrated in the case of the systems of interest to us here— is that the linearized system, contrary to the initial system, may not be controllable. The usefulness of such an approximation is then rather limited. Yet, the aforementionned result ensures the existence of another homogeneous approximation, with respect to another dilation, which is controllable and thus a better local approximation of the system. This approximation, although nonlinear, is polynomial. This already represents a major simplification with respect to the general case.

The above considerations explain why the homogeneous approximation of a control system represents a useful extension of the more familiar concept of linear approximation. Moreover, there exist also several results —generalizing known results associated with linear approximations— which specify the conditions under which a control feedback designed for the homogeneous approximating system allows to (locally) stabilize the equilibrium point of the original system. Consider for instance, a state feedback control $u(x)$ which, when applied to a controllable homogeneous approximation of some system, i) gives a closed-loop system which is still homogeneous (i.e. such that the field $f$ of the closed-loop system $\dot{x} = f(x)$ is homogeneous), and ii) yields local asymptotic stability of the equilibrium $x = 0$ of this homogeneous system. Then, this feedback control also ensures local asymptotic stability of $x = 0$ for the original system. Moreover, if the degree of homogeneity of the closed-loop approximation is zero, then the convergence to zero of the system's solutions is necessarily (locally) uniformly exponential —as in the case of a linear feedback which stabilizes a linear approximation of a nonlinear system.

## 5. Motion, controllable linear approximations, and application to control design

We have seen that the linear approximation of system (1), at an equilibrium point $(x = x^\star, u = 0)$, can be written as

$$\dot{x} = B(x^\star)u \tag{9}$$

with $B(x^\star) = (b_1(x^\star) \cdots b_m(x^\star))$, and that this approximation is not stabilizable. As a consequence, classical control techniques, for linear systems or nonlinear ones which can be linearized via a change of coordinates and/or static or dynamic feedback, do not apply, at least directly, to the problem of asymptotic stabilization of a *fixed* point $x = x^\star$. A theorem, due to Brockett [3], implies moreover that no continuous state feedback $u(x)$ can make this point asymptotically stable. Nonetheless, one must not hastily conclude from the above results that classical linear or nonlinear control techniques are never applicable to these systems. *It all depends on the pursued control objective*. Asymptotic stabilization of a system's equilibrium point may be the objective of interest. But it is by no means the only possible one. This remark will be illustrated in the next three sub-sections.

### 5.1. Stabilization of a partial state vector

In some cases the task allocated to the vehicle does not require active control or stabilization of all configuration variables. This may have important consequences on the control design, and also yield significant simplifications. Let us take the example of the three-dimensional chained system (8), used to model a unicycle-type vehicle, and assume that only the values of the two variables $x_1$ and $x_2$ are of interest for the application. This means that all underlying control problems bear only upon the *reduced* state $x_{red} = (x_1, x_2)^T$ and the corresponding subsystem

$$\dot{x}_{red} = u$$

with $u = (u_1, u_2)^T$. This particularly simple subsystem is linear, time-invariant, and controllable. Its control thus relies on the classical framework of linear control theory.

In practice, the reduced state vector of interest is usually different from the one considered above. However, this is of no importance as far as it is possible to end up with a controllable linear subsystem. Consider for instance the application consisting of controlling the position of one point $Q$ attached to the vehicle, and assume that this point is located on the longitudinal central axis, at a (signed) distance $d$ from the rear axle, i.e.

$$\overrightarrow{PQ} = d(\cos x_3 \vec{\imath} + \sin x_3 \vec{\jmath})$$

Let $x_{Q,1}$ and $x_{Q,2}$ be the coordinates of this point in the fixed frame attached to the vehicle's plane of motion. We can regroup them in the reduced state vector $x_{red} \triangleq (x_{Q,1}, x_{Q,2})^T$. It is simple to derive the following equation from the kinematic model (2)

$$\dot{x}_{red} = R(x_3)u\,, \qquad R(x_3) \triangleq \begin{pmatrix} \cos x_3 & -d\sin x_3 \\ \sin x_3 & d\cos x_3 \end{pmatrix}\,, \quad u \triangleq (u_1, u_2)^T$$

By setting $v \triangleq R(x_3)u$, we recover the former time-invariant linear equation. However, this is of interest only when the matrix $R(x_3)$ is invertible – in order to be able to determine the physical inputs $u_1$ and $u_2$ from the vector $v$ –, i.e. when $d \neq 0$. Of course, no control derived in this manner can *a priori* guarantee anything upon the dynamics of the orientation variable $x_3$. The analysis of this variable's asymptotic behavior belongs to the study of the system's *zero dynamics*. We will give, a little further within the context of *path following with orientation servoing*, another application example which involves a reduced state vector and may be brought back to a linear control problem, provided that an assumption of "persistent" motion is met.

### 5.2. Trajectory stabilization

Asymptotic stabilization of an equilibrium point of a driftless system is just in fact a particular case of the problem of *asymptotic stabilization of solutions/trajectories* of the system. This more general problem can be formulated as follows. Let $x_r(t)$ $(t > 0)$ denote a system's solution – called a *reference* solution – obtained by applying a specific reference control input $u_r(t)$. By definition, this solution satisfies, at every time-instant, the equation

$$\dot{x}_r \;=\; B(x_r)u_r \tag{10}$$

Let us define $\widetilde{x} \triangleq x - x_r$, and $\widetilde{u} \triangleq u - u_r$. Then the point $(\widetilde{x} = 0, \widetilde{u} = 0)$ is an equilibrium of the *error* system defined by

$$\dot{\widetilde{x}} \;=\; B(\widetilde{x} + x_r(t))(\widetilde{u} + u_r(t)) \;-\; B(x_r(t))u_r(t) \tag{11}$$

The problem consists in determining a state feedback $\widetilde{u}(\widetilde{x}, t)$ which asymptotically stabilizes the origin $\widetilde{x} = 0$ of the error system (11).

In this way, the problem of stabilizing an arbitrary solution of system (1) has been brought back to the one of stabilizing the origin of a “related”, but usually different, error system (11). The fact that the problem of asymptotic stabilization of the origin of system (1) cannot be solved by classical control techniques does not imply that the same holds true for the asymptotic stabilization of the origin of the error system (11). Let us illustrate this by first determining the linear approximation of the error system at the point $(\widetilde{x} = 0, \widetilde{u} = 0)$. We obtain

$$\dot{\widetilde{x}} \;=\; A(t)\widetilde{x} \;+\; B(x_r(t))\widetilde{u}\,, \qquad A(t) \triangleq \sum_{i=1}^{m} \frac{\partial b_i}{\partial x}(x_r(t))u_{r,i}(t) \tag{12}$$

The main difference between this equation and the one of the linear approximation (9), that we have considered initially, is the *drift* term $A(t)\widetilde{x}$. This term is identically equal to zero when the reference input $u_r(t)$ is itself identically equal to zero. Otherwise, a complementary analysis is required to determine whether the linear system (9) is, or is not, state feedback stabilizable. Let us, for example, perform this analysis for the unicycle model (6) when, for every $t \geqslant 0$, $x_{r,i}(t) = 0$, $\forall i \in \{2, 3\}$ and $u_{r,2}(t) = 0$ (one easily verifies that such a solution corresponds, in practice, to having the unicycle move along a straight line). A simple calculation gives

$$A(t) = \frac{\partial b_1}{\partial x} u_{r,1}(t)\;, \quad \text{with} \quad \frac{\partial b_1}{\partial x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{13}$$

and

$$B(x_r(t)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{14}$$

If we further assume that $u_{r,1}$ is constant (meaning that the vehicle's longitudinal velocity is constant) it is simple to verify – by application of the classical controllability criterion in the linear case – that the obtained system is always controllable, except in the case where $u_{r,1} = 0$. This example illustrates well that non-stabilizability of the linear approximation of a system in the form (1) does not imply by itself that classical linear control techniques do not directly apply, even when the regulation problem bears upon the complete state of the system. It depends on the control objective, and thus on the nature of the application. In the context of mobile robotics, this example also suggests – as this has been shown in more complete related works [9–11, 35] – that control objectives which involve *non-intermittent motion* of the vehicle can usually been solved satisfactorily by applying linear control techniques. Using more complex methods, such as exact (static or dynamic) state feedback linearization, is justified only in a small number of cases.

The fact that the linear approximation of the trajectory tracking error system becomes uncontrollable when $u_{r,1} = 0$ points out a difficulty which the control design must take into account. For instance, it is tempting to try enforcing, for the closed-loop system, a pole placement which is invariant and independent of the "velocity" $u_{r,1}$. This would be a mistake because it would yield a state feedback with a gain matrix norm growing unbounded when $u_{r,1}$ tends to zero and, therefore, a control which is not defined when $u_{r,1} = 0$. It is better to consider a *varying pole placement* strategy which depends on the size of $|u_{r,1}|$ in a way to yield a control which vanishes (whatever the measured tracking error) when $u_{r,1}$ gets equal to zero. Indeed, what could be the value of nonzero control action when the model on which the control design is based is not stabilizable? There are various ways of implementing such a varying pole placement: minimization of a quadratic cost (LQ control) with increasing penalty on the control energy when $|u_{r,1}|$ decreases, determination of precalculated control gains for a discrete set of values of $u_{r,1}$ complemented with an interpolation procedure for the intermediary values, etc... A simple and systematic way of implementing a ponderation on the control objective, depending on the size of $|u_{r,1}|$, consists in a *time scaling* based on the introduction of a "new" time variable as defined by

$$s \;:\; t \mapsto s(t) = \int_0^t |u_{r,1}(v)|dv \tag{15}$$

In the new time frame measured by the variable $s$ – equal to the length of path travelled by the reference vehicle –, the linear approximation of the error system (12) becomes, by setting $\widetilde{x}' = \frac{d\widetilde{x}}{ds}$ and $\widetilde{u}' = \frac{\widetilde{u}}{|u_{r,1}(t)|}$,

$$\widetilde{x}' \;=\; \frac{\partial b_1}{\partial x}\widetilde{x}\; sign(u_{r,1}(t)) + B\widetilde{u}' \tag{16}$$

During the time intervals when $u_{r,1}(t)$ has a constant sign, the above equation is similar to the equation of a simple controllable time-invariant linear system whose expression does not depend upon the size of $|u_{r,1}(t)|$. Let us assume, for instance, that this sign is positive. Let $\widetilde{u}' = K^+\widetilde{x}$ denote a linear state feedback control which exponentially stabilizes (with respect to the new variable $s$) the origin of this system. The state feedback control law obtained with this method is then

$$u = \begin{pmatrix} u_{r,1}(t) \\ 0 \end{pmatrix} + K^+\widetilde{x}|u_{r,1}(t)|$$

Similarly, in the case where the sign of $u_{r,1}(t)$ is negative, we obtain $\widetilde{u}' = K^-\widetilde{x}$. The two control laws so obtained connect well together (by continuity) when $u_{r,1}$ passes through zero – since they are both equal to zero in this case –, and it is not difficult to find out conditions upon $u_{r,1}(t)$ that (theoretically) guarantee local asymptotic stabilization of the error system's origin. It suffices, for instance, that the sign of $u_{r,1}(t)$ be constant and that $s(t)$ tend to infinity when $t$ itself tends to infinity.

**Remarks :**

1. In the framework of mobile robot control, the time-scaling technique considered above, whose application does not impose to work on a linear approximation of the error system, has been used in several contributions, e.g. [9–11, 35]. One can trace its origin back to [28, 43].
2. The extension of the control approach described above to the case of an arbitrary reference trajectory is done naturally by expressing the tracking error (in position and orientation) in the basis of a mobile frame attached to the reference vehicle.

### 5.3. Path following with active servoing of the vehicle's orientation

A problem related to trajectory stabilization, that can be solved by using similar control design techniques, is the problem of *path following*. In this case, a curve (or path) in the vehicle's plane of motion, as well as the longitudinal velocity of the vehicle, are prespecified. The assumption on the vehicle's longitudinal velocity means, for instance, that a separate control action takes care of the regulation of this velocity so that the corresponding variable is no longer free and available for control purposes. We are thus left with a single remaining control input variable (the vehicle's body angular velocity, in the case of a unicycle-type vehicle, and the angular velocity of the driving wheel, in the case of a car-like vehicle) in order to ensure asymptotic stabilization to zero of i) the distance between one point attached to the vehicle and its "projection" on the chosen curve, and ii) the angle corresponding to the difference in orientation between the vehicle and the tangent to the curve at the projection point. Automated road following by a car is an application of this problem. Indoors wall following by a mobile robot is another one. Note that in these two examples the path to be followed is "given by the environment" (by opposition to the result of some calculation). On one hand, this may be interpreted as a simplification (no need to compute a path). On the other hand, this may also represent a source of difficulties because the geometry of such a path may not be known in advance (precisely because it is not the result of a planification).

The second regulation objective, which bears upon the vehicle's orientation, is of real interest only when travelling along the path is done with either positive (forward motion) or negative (backward motion) longitudinal velocity and no drastic change in orientation must result from a change of sign of this velocity. Indeed, in the case where the vehicle's orientation does not matter, we have already seen, on the unicycle's example, that it was possible to address the path following problem via the positioning of a point attached to the vehicle (not located on the actuated wheels' axle) and by considering an adequate reduced system which is linear and controllable. For the tricycle (or car) example this strategy is easily adapted by considering a point rigidly linked to a frame attached to the driving wheel, provided that it is not on the vertical axis of the wheel.

Let us return to the example of the unicycle modeled by (2) and assume, to simplify, that the followed path is the fixed frame's axis corresponding to the coordinate $x_1$. As long as the vehicle's orientation angle, with respect to this axis, remains in the domain $(-\pi/2, +\pi/2)$, it is simple to verify that the problem is equivalent to stabilizing the variables $x_2$ and $x_3$ of the three-dimensional chained system to zero by using the sole control input $u_2$ (the input $u_1(t)$ being given by assumption). By setting $x_{red} = (x_2, x_3)^T$, we are thus brought back to the stabilization of the origin of the following reduced system

$$\dot{x}_{red} = \begin{pmatrix} 0 & 0 \\ u_1(t) & 0 \end{pmatrix} x_{red} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_2 \tag{17}$$

This system is linear and, moreover, time-invariant and controllable when $u_1$ is constant and different from zero. As in the path tracking case, one can transform (17) into a time-invariant controllable system by introducing a time-scaling such as (15), with $u_1(t)$ playing now the role of $u_{r,1}(t)$. We leave to the interested reader the tasks of working out the details of the control design and verifying that the method can be generalized to any curve, via the introduction – as done for example in several works based upon [45] – of a mobile *Frénet frame* sliding along the curve and a corresponding set of coordinates used to parametrize the vehicle's position with respect to this frame.

**Remark:** It may be useful to recall (since this point is often omitted in research articles dealing with nonlinear systems) that it is sometimes quite useful to complement the control law with an integral action in order to compensate for slow additive perturbations (which can be modeled by constant terms) acting on the system, and also biases which may be present in the measurement of the state variables. For instance, in the case of path following, incorporation in the control law of an integral of the distance between the vehicle and the path will help in zeroing this distance when a biased measurement of the vehicle's orientation angle is used as an estimation of this angle.

## 6. Time-varying stabilization of a fixed point

Control techniques reviewed in the previous section do not allow to solve the problem of asymptotic stabilization of an equilibrium point of (1) nor, therefore, the one of asymptotic stabilization of a desired position/orientation for a nonholonomic wheeled vehicle. Moreover, in view of Brockett's theorem, it is pointless to look for a continuous state feedback control depending on the system's state only. A more recent result by Coron and Rosier [7], using the notion of solution *in the sense of Filippov* to a differential equation whose righthand term is discontinuous, also suggests that state feedbacks that are discontinuous at the point to be stabilized cannot be stabilizers. Several "false" solutions to the problem of asymptotic stabilization of an equilibrium point have been proposed in the literature. They are false in the sense that they do not ensure stability (in the classical sense of Lyapounov). Now, this does not mean that they are devoided of any practical interest. In particular, nothing prevents *a priori* such controls from being useful, or even effective, when the problem consists of "getting near" the equilibrium point of interest. This problem may also be addressed by planning a trajectory, parametrized by the time index and ending at the chosen equilibrium point, and considering any trajectory stabilization technique for the design of a feedback control (see previous section). This kind of strategy is perfectly admissible and sensible in practice. But one should realize that it can by no means pretend solving the problem of asymptotic stabilization in the sense of Lyapounov. Ultimately, it is the user who has to determine whether this feature is, or is not, important for his application. In order to work out an adequate control strategy, possibly consisting of switching between several control laws, it matters that the user knows about solutions which theoretically yield asymptotic stability. Such solutions exist, but they have to belong to a class of feedbacks larger than the one traditionally considered. An example of such a class is the set of continuous functions depending not only on the system's state $x$ but also on an exogenous variable such as the time variable $t$ – with this latter dependence giving rise to the *time-varying* denomination.

### 6.1. "Slow" Lipschitz continuous stabilizers

For the set of systems of interest to us here (which are not stabilizable by continuous "pure" state feedback), the first continuous time-varying state feedback ensuring asymptotic stability of the origin of a mono-chained system with two control inputs has been proposed – to our knowledge – in [44]. For instance, in the case of a three-dimensional chained system, such a feedback control is

$$\begin{aligned} u_1 &= -k_1 x_1 + k_\omega \sin(\omega t) x_3 \\ u_2 &= -k_2 x_2 - k_3 u_1 x_3 \end{aligned} \tag{18}$$

with $k_1 > 0$, $k_2 > 0$, $k_3 > 0$, $k_\omega \neq 0$, and $\omega \neq 0$. The proof of this result – (global) asymptotic stabilization of the origin – is a simple exercise of analysis which uses the decrease of the function $V : (x_2, x_3) \mapsto V(x_2, x_3) \stackrel{\Delta}{=} x_2^2 + k_3 x_3^2$ along every solution to the closed-loop system. It is useful to go through this proof in order to understand the role played by the parameters $\omega$ and $k_\omega$ which account for the time-varying nature of the control.

The time-varying feedback (18) is well defined everywhere. Computing its value, for any given state vector and time instant, is straightforward since it is infinitely differentiable, and even analytic, at the point that we wish to stabilize. However, it has an important shortcoming, shared in fact by all continuous state feedbacks that are Lipschitz at the stabilized equilibrium point: it only yields “slow” convergence to the origin in general, a feature which is little conform to the performance capability suggested by the open-loop controllability property. For example, in the case of the control (18), the rate of convergence is usually – for most initial conditions – not better than polynomial, like $t^{-\frac{1}{2}}$. This implies in particular that the length of the path travelled to get arbitrarily close to the origin tends to infinity.

### 6.2. Homogeneous “exponential” stabilizers

Methods have been developed for the design of time-varying continuous state feedbacks which yield uniform exponential convergence – a usual objective in the linear case. To obtain this rate of convergence, feedback laws that make the closed-loop system’s vector field – i.e. the term $f(x,t)$ in the equation $\dot{x} = f(x,t)$ of this system – *homogeneous of degree zero* (with respect to some arbitrary dilation) can be designed. The exponential convergence rate associated with a linear system whose origin is asymptotically stable is itself a consequence of this property of homogeneity. Several control design methods have been proposed. One of them consists in modifying – one could say “homogeneize” – a stabilizing control, itself obtained by applying any other design technique [27]. Another provides a general design algorithm applicable to any driftless system, by using averaging properties associated with high-frequency periodic terms [30]. A third one exploits the specific structure of chained systems and allows to meet specified performance objectives independently of the frequencies used in the periodic terms entering the control expression [33].

An example of such a homogeneous periodic feedback in the case of the three-dimensional chained system, which can be interpreted as a modified version of the differentiable feedback (18), is

$$\begin{aligned} u_1 &= -k_1 x_1 + \omega \sin(\omega t) \rho(x_2,x_3)^{\frac{1}{2}} \\ u_2 &= -k_2 |u_1| \frac{x_2}{\rho(x_2,x_3)^{\frac{1}{2}}} - k_3 u_1 \frac{x_3}{\rho(x_2,x_3)} \end{aligned} \qquad (19)$$

with $k_i > 0$ ($i = 1,\ldots,3$), $\omega \neq 0$, and $\rho(x_2,x_3)$ the real positive root of the polynomial of degree three $P(\alpha,x_2,x_3) = \alpha^3 - \alpha\frac{x_2^2}{k_3} - x_3^2$. One easily verifies that i) the positive function $\rho$ is homogeneous of degree two with respect to the dilation $\Delta_\lambda^r x = (\lambda x_1, \lambda^2 x_2, \lambda^3 x_3)$, ii) the closed-loop system is homogeneous of degree zero with respect to this dilation, and iii) the feedback control (19) is defined everywhere, in particular at points where $\rho(x_2,x_3) = 0$, i.e. such that $x_2 = x_3 = 0$, by the sake of continuity. However, this feedback is not Lipschitz at these points. This already induces a difficulty as for the numerical calculation of the input $u_2$ in the neighborhood of these points, as well as much sensivity against measurement noise upon the variables $x_2$ and $x_3$. These difficulties have their origin in the specific structure of the system. They arise, in the case of wheeled vehicles, from physical nonholonomic linkages and thus cannot be avoided.

### 6.3. The robustness issue

The extreme sensitivity of homogeneous stabilizers in the neighborhood of the origin underlies a real practical weakness whose consequence, analyzed in [25], is a defect of robustness against modeling errors. More precisely, it is proved that an arbitrarily small error on the modeling of a control vector field $b_i$ of the system (1) can make the origin of the closed-loop system (locally) unstable. For instance, the origin of the system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1 x_2 + \epsilon u_1 , \end{aligned} \tag{20}$$

with the homogeneous feedback control (19) applied to this system, is unstable whatever $\epsilon \neq 0$ – whereas it is asymptotically stable, with uniform exponential convergence of the solutions to the origin, when $\epsilon = 0$. Typically, trajectories no longer converge to the origin but tend to a limit cycle in the neighborhood of this point. This defect of robustness contrasts with the case of linear systems. Indeed, a fundamental robustness property of linear systems is that any exponential (thus homogeneous) feedback stabilizer of an equilibrium point still ensures exponential stability of this point when modeling errors upon the system's state and/or control matrices are small enough. A few attempts [2, 31] to by-pass this defect involve periodic homogeneous feedbacks, different from those evoked previously in that they depend upon periodically sampled values of the system's state vector (by opposition to continuous-time dependence). One can then speak of a *hybrid continuous/discrete* feedback control, with the word "continuous" indicating that the control inputs, at the time-instants when they are updated, are continuous functions of the state, whereas the word "discrete" stands for the fact that the control is updated with respect to the state only periodically. An example of such a feedback, for the three-dimensional chained system, is

$$\begin{aligned} u_1(x(kT), t) &= \frac{1}{T}\left[-x_1(kT) + 2\pi |x_3(kT)|^{1/2} sign(x_3(kT)) \sin(\omega t)\right] \\ u_2(x(kT), t) &= \frac{1}{T}\left[-x_2(kT) - 2|x_3(kT)|^{1/2} \cos(\omega t)\right] \end{aligned} \tag{21}$$

for $t \in [kT, (k+1)T)$, $k \in \mathbb{N}$, $\omega > 0$, and $T = \frac{2\pi}{\omega}$ denoting the control update period. Contrary to the continuous/continuous feedback (19), this control ensures stability of the origin of system (20) whatever the size of $|\epsilon|$. It also yields convergence of solutions to zero in finite time, but this is a particular case. More generally, the algorithm proposed in [31] for the construction of hybrid feedbacks applies to any controllable driftless system and yields a rate of convergence at least exponential. This type of solution thus seems endowed with good properties. However, it appears at a closer look that the problem of robustness has, in fact, just been displaced. Indeed, computerized implementation of the control (21) implies transforming the control law into a piecewise constant one. The simplest method consists in using, as commonly done in the linear context, a zero-order hold (ZOH). In order to keep the properties of asymptotic stability and robustness against modeling errors, the ZOH sampling frequency must be chosen equal to a multiple value of the updtating frequency $1/T$. Let $\Delta$ denote the sampling frequency. In practice, the feedback control so obtained is never implemented as such exactly, would it be only because of delays resulting from computations or communications. A way of modeling some of the implementation imperfections consists in assuming that the control sampling frequency slightly fluctuates around its central value $\Delta$. It is not difficult to show that a fluctuation of this kind has no important consequences in the case of a linear system stabilized by a sampled linear feedback. In particular, the property of asymptotic stability is preserved – an argument definitely in favor of linear feedback control versus other controls which ideally yield finite time convergence. On the other hand, simulation of the control (21) will show the loss of stability of the origin, however small (but different from zero) the size of the fluctuation. This illustrates the sensitivity and a type of lack of robustness of this control in practice.

To summarize what precedes:

- the problem of asymptotic stabilization of a controllable driftless system (whose dimension exceeds the number of control inputs) has solutions, one of which is time-varying continuous feedback control,
- if this control is Lipschitz (at the point to be stabilized), it can ensure robustness of the property of asymptotic stability [26] but is not compatible with the requirement of fast (at least exponential) convergence,
- if this control is not Lipschitz but yields a closed-loop system which admits a homogeneous approximation of degree zero – the control may be of type continuous/continuous or hybrid continuous/discrete –, it ensures exponential convergence, but not the robustness of the stability property.

This indicates that finding a satisfactory practical compromise between rapidity (of convergence) and robustness (of the stability property) might be impossible. This in turn suggests that, for driftless nonlinear systems, asymptotic stabilization of an equilibrium point may not constitute, contrary to the case of linear systems, "the" basic control problem of interest.

## 7. Complementary remarks about trajectory stabilization

Let us consider the context of Robotics and come back to the problem of tracking a reference vehicle, knowing (see section 5.2) that this problem can be formulated in terms of trajectory stabilization. We have shown that under certain conditions, bearing upon the "persistence" of the vehicle's motion in particular, classical control methods –

of Linear Control Theory, for instance – are applicable and (theoretically) allow to ensure convergence to zero of the tracking error. These conditions are not satisfied when the reference vehicle is motionless. In this case, the tracking problem becomes asymptotic stabilization of the (fixed) position/orientation of the reference vehicle. We have pointed out solutions to this latter problem in the form of time-varying continuous feedbacks. We have also commented upon the structural difficulty – which we suspect to be an impossiblity – of finding a satisfactory rapidity/robustness compromise. Which strategy should then one adopt when the motion of the followed vehicle is unknown in advance? In particular, is it necessary to switch from one control law to another depending on whether the reference vehicle seems to be moving or almost stopped? There does not exist, to our knowledge, extensive and detailed studies of these (difficult) questions. In order to avoid switching between several control laws, one could have hoped for the existence of feedbacks, possibly time-varying and depending on both position error between the two vehicles and velocity of the reference vehicle, which would have uniformly ensured asymptotic stabilization of the tracking error to zero. This would have solved the problem, at least in theory. For instance, recall that in the case of a controllable linear system of equation $\dot{x} = Ax + Bu$ *every* reference trajectory $x_r(t)$ solution to this system (thus such that $\dot{x_r} = Ax_r + Bu_r$) is asymptotically stabilized by the feedback control $u(x - x_r(t), u_r(t)) \stackrel{\Delta}{=} u_r(t) + K(x - x_r(t))$, with $K$ denoting a matrix such that the matrix $(A + BK)$ is Hurwitz stable (e.g. with eigenvalues in the left-half complex plane). Unfortunately, it is essentially proved in [24] that a solution of this type cannot exist in the case of a chained system, as subsequently in the case of any vehicle whose kinematics can be modeled by a chained system. The proof of this result precisely relies on the construction, for any feedback belonging to the considered class, of a reference trajectory which is not stabilized. In other words, whatever the chosen control law, there exists a specific motion for the reference vehicle which prevents stabilization of the tracking error to zero. Should one deduce from this fact that a complete study of the vehicle tracking problem necessarily involves switching control strategies?

## 8. Practical stabilization via the use of periodic transverse functions

In this section, we try to bring a few answers to the questions asked before. The basic difficulty is that, for driftless systems, the objective of uniformly fast and robust asymptotic stabilization of arbitrary reference trajectories does not seem reachable. This objective, on which much of the control theory of linear (or weakly nonlinear) systems is built, thus seems ill-adapted to this class of nonlinear systems. Its relaxation into another objective, more representative of the possibilities offered by these systems, seems also a reasonable way of addressing control issues with a new perpective. The control objective which will be considered in what follows corresponds to a type of *practical stabilization* with the aim of stabilizing a neighborhood – arbitrarily small – of the origin of the associated error system. Such an approach is not only useful conceptually. It corresponds also to the practical experience of drivers of nonholonomic wheeled vehicles (car, crane,...) that positioning such a vehicle with extreme precision is neither simple, nor required in nominal situations. Whenever a strong positioning constraint is imposed by the application, it is usually preferable to rely upon a holonomic manipulator arm, better adapted to this goal and simpler to control. This manipulator can be mounted on the nonholonomic vehicle in order to combine the advantages of both mechanical structures – one then speaks of a *mobile arm*. Still, it is often necessary to make "maneuvers" in order to transfer the vehicle from some initial configuration to a closeby one. This is performed via a sequence of forward and backward motions which serve to "mimic" the displacement of a vehicle capable of instantaneous motion in every direction. In other words, it matters to approximate a trajectory which cannot be followed exactly. We will see how the proposed concept of practical stabilization applies naturally to this type of goal and simplifies the design of a class of differentiable feedback controls meeting newly fixed objectives.

### 8.1. Unicycle example

Let us consider the chained system (8) of dimension three, and substract from the state vector $x$ the vector

$$f_\epsilon(\theta) \stackrel{\Delta}{=} \left(\epsilon_1 \cos\theta, \epsilon_2 \sin\theta, \frac{\epsilon_1\epsilon_2}{4}\sin 2\theta\right)^T \tag{22}$$

with $\epsilon_1 > 0$ et $\epsilon_2 > 0$, so as to obtain the vector $y \stackrel{\Delta}{=} x - f_\epsilon(\theta)$. The time derivative of this vector, along the chained system's solutions, is given by the following equation

$$\dot{y} \;=\; B(x,\theta)\, U_e$$

with

$$B(x,\theta) = \begin{pmatrix} 1 & 0 & \epsilon_1 \sin\theta \\ 0 & 1 & -\epsilon_2\cos\theta \\ x_2 & 0 & -\frac{\epsilon_1\epsilon_2}{2}\cos 2\theta \end{pmatrix}$$

and $U_e \stackrel{\Delta}{=} (u_1, u_2, \dot{\theta})^T$, a vector which may be interpreted as an "extended" control vector, with $\dot{\theta}$ as a complementary control variable.

Since the norm of $f_\epsilon(\theta)$ can be made arbitrarily small, independently of the value of $\theta$, via the choice of the parameters $\epsilon_1$ and $\epsilon_2$, it suffices to asymptotically stabilize $y$ to zero to make $x(t)$ become small when $t$ tends to infinity. The equation of evolution of $y$ suggests considering the following control

$$U_e \;=\; B(x,\theta)^{-1} K y$$

with $K$ denoting a stable matrix, so as to obtain the closed-loop system $\dot{y} = Ky$ and, subsequently, the exponential stability of $y = 0$.

This solution would be particularly simple if it were defined everywhere, i.e. if the matrix $B(x,\theta)$ were always invertible. It is in fact simple to verify that it is invertible at every point $x = f_\epsilon(\theta)$. This property indicates that any variation of the function $f_\epsilon$ at a point $\theta$ is *transverse* to the directions given by the vector fields $b_1$ et $b_2$ evaluated at the point $f_\epsilon(\theta)$. We will just say, by a shortcut of language, that the function $f_\epsilon$ is *transverse* to the vector fields $b_1$ and $b_2$. It is equally simple to verify that the matrix $B(x, \theta)$ is not invertible, for instance, at $x = 0$ when $\theta = \pi/4 + k\pi/2$ ($k \in \mathbb{Z}$). While it is possible to show that the above defined state feedback indeed stabilizes $y = 0$ asymptotically when the initial condition $y(0) = x(0) - f_\epsilon(\theta(0))$ has its norm small enough, this result is not true for other initial conditions, since the control is not even defined at some points. In order to obtain a more satisfactory solution – globally well defined, in particular – one has to introduce a change of coordinates slightly more involved than the one considered above. Such a change of coordinates is, for instance

$$z = \begin{pmatrix} x_1 - f_{\epsilon,1}(\theta) \\ x_2 - f_{\epsilon,2}(\theta) \\ x_3 - f_{\epsilon,3}(\theta) - x_1(x_2 - f_{\epsilon,2}(\theta)) \end{pmatrix} \tag{23}$$

One easily verifies that the (uniform exponential) convergence of $z$ to zero yields the one of $x - f_\epsilon(\theta)$, independently of the values taken by $\theta$. Furthermore, differentiation of $z$ with respect to time yields the following equation

$$\dot{z} = D(x_1)B(f_\epsilon(\theta), \theta)\, U_e \tag{24}$$

with

$$D(x_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -x_1 & 1 \end{pmatrix}$$

Since the matrices $D(x_1)$ and $B(f_\epsilon(\theta), \theta)$ are always invertible, the state feedback

$$U_e = B(f_\epsilon(\theta), \theta)^{-1} D(x_1)^{-1} K z \tag{25}$$

with $K$ denoting a stable matrix (chosen in relation to the desired rate of convergence), is well defined at any point and yields, for the closed-loop system, *global* exponential stability of $z = 0$ and, subsequently, of $x - f_\epsilon(\theta) = 0$. In view of the expression (25), it follows also that the "oscillation frequency" $\dot{\theta}(t)$ tends to zero. Since this convergence is exponential, the angle $\theta(t)$ converges to a constant number. Therefore, it suffices to choose the parameters $\epsilon_1$ and $\epsilon_2$ small to obtain exponential convergence of $x(t)$ to a point located near zero.

In fact, the potentialities of the control design approach considered above appear fully when considering the case of an additive (measurable) perturbation $p(x, t)$ acting on the system. The system's equation is then

$$\dot{x} = b_1(x)u_1 + b_2 u_2 + p(x, t) \tag{26}$$

and it is simple to verify that the following modified control

$$U_e = B(f_\epsilon(\theta),\theta)^{-1}\left(D(x_1)^{-1}Kz - \begin{pmatrix} p_1 \\ p_2 \\ p_3 - p_1(x_2 - f_{\epsilon,2}(\theta)) \end{pmatrix}\right) \tag{27}$$

allows to exactly "compensate" for the perturbation $p$, and yields the same equation $\dot{z} = Kz$ of the closed-loop system. It follows that uniform exponential convergence of $x - f_\epsilon(\theta)$ to zero is again obtained. On the other hand, the control vector and, subsequently, the oscillation frequency $\dot{\theta}(t)$, may not tend to zero in this case, depending on the specific characteristics of the perturbation.

A direct application of the above control concerns the problem of tracking a reference vehicle – the reader interested in this subject may consult the article by Dixon *et al.* [13] which inspired us the control approach here described –, and also the extension of this problem to the case where the reference vehicle is free to move in every direction while the controlled vehicle remains submitted to a nonholonomic constraint forbidding instantaneous lateral motion. It is not difficult to show (see [34], for example) that the error system associated with this problem is in the form (26). What precedes allows us to infer that the control (27) "unconditionally" guarantees practical tracking of the reference vehicle with a tracking error which, although it does not converge to zero, can be made as small as desired via the choice of the parameters $\epsilon_1$ and $\epsilon_2$.

Another interpretation of this result is that the feedback control so defined allows to approximate, with arbitrarily good precision, any unconstrained trajectory by an admissible (feasible) trajectory of the nonholonomic unicycle. Application to the path-planning problem, with obstacles avoidance, then appears clearly since one of the steps in solving this problem traditionally consists in approximating an initial trajectory which does not take the nonholonomic constraint into account. Practical modalities of the approach need of course to be further studied and developed.

### 8.2. Generalization to $n$-dimensional chained systems

The control technique introduced in the previous section in the case of the three-dimensional chained system can be generalized to any chained system (8) by first defining a periodic function $f_\epsilon : \theta \in \mathbb{R}^{n-2} \mapsto f_\epsilon(\theta) \in B^n(0,\epsilon)$, with $B^n(0,\epsilon)$ a closed ball in $\mathbb{R}^n$ centered at zero and of radius $\epsilon$, such that the matrix

$$B(x,\theta) = \left(b_1(x), b_2, -\frac{\partial f_\epsilon}{\partial \theta_1}(\theta), \ldots, -\frac{\partial f_\epsilon}{\partial \theta_{n-m}}(\theta)\right)$$

is invertible at the point $x = f_\epsilon(\theta)$, for any $\theta$. The existence of such a function – the fundamental point that allows to generalize the control approach illustrated by the preceding example – is a consequence of the property of controllability of the chained system. An even more general version of this result states the equivalence between the satisfaction of the Lie Algebra Rank Condition (LARC) for a set of vector fields $\{b_1, \ldots, b_m\}$ and the existence of bounded periodic functions $f_\epsilon$ transverse to these vector fields [32]. Eventhough the proof of this result is constructive, the algorithm used for the construction of transverse functions is, in general, rather complicated. However, in the specific case of chained systems, a simple iterative construction is given in [34]. The generalization of relation (23) which defines the vector $z$ is, in this case

$$z = D(x_1)(x - f_\epsilon(\theta))$$

with $D(x_1) = \exp(-Ax_1)$, $A = \frac{\partial b_1(x)}{\partial x}$ (a constant matrix). The time derivative of $z$ along an arbitrary solution to the perturbed chained system (26) is then given by

$$\dot{z} \;=\; -Azp_1 + D(x_1)\left[B(f_\epsilon(\theta),\theta)U_e + p\right] \tag{28}$$

with $U_e \stackrel{\Delta}{=} (u_1, u_2, \dot{\theta}_1, \ldots, \dot{\theta}_{n-2})^T$, an extended control vector composed of the initial control variables $u_1$ and $u_2$ complemented with $n-2$ *oscillation frequencies*. Global exponential stability of $z = 0$ is then simply obtained by considering, for instance, the state feedback

$$U_e \;=\; B(f_\epsilon(\theta),\theta)^{-1}\left(-p + D(x_1)^{-1}Azp_1 + D(x_1)^{-1}Kz\right) \tag{29}$$

with $K$ denoting a stable matrix. We show in [36] how this approach can be generalized to any controllable homogeneous driftless system.

### 8.3. Comparison with other control methods

For mobile robot applications, such a comparison will remain speculative until the new control approach is tested on physical systems. It is nevertheless possible to point out characteristics which, conceptually at least, seem of interest with respect to other control methods evoked earlier in the article.

- First, this control is not meant to stabilize a desired position/orientation precisely. The aim is to stabilize a (small) neighborhood of such a configuration. This may be seen as a weak point, in particular with respect to other control methods which provide – at least in theory – solutions to the point stabilization problem. We believe that control design efforts should not focus on this issue, for the highly nonlinear systems considered here. Indeed, the study of the performance/robustness compromise associated with the problem of asymptotic stabilization of an equilibrium point strongly suggests that meeting the properties which are taken for granted when a stabilizing linear feedback is applied to a linear system is a hopeless objective for these nonlinear systems. As a complementary comment, it should be noted that the development of this control approach – based on the use of periodic transverse functions – is still in its infancy so that one cannot exclude the possibility of future modifications (extensions) yielding new (theoretical) solutions to the aforementionned stabilization problem.

- Control laws obtained with this approach are related to time-varying feedbacks proposed in the literature as solutions to the problem of asymptotic stabilization of an equilibrium point. Indeed, $\dot{\theta}$ may be interpreted as a vector of instantaneous oscillation frequencies which, if it were constant, would yield a smooth periodic state feedback. Explicit use of these frequencies as (independent) auxiliary control variables allows to increase the dimension of the control vector up to the original system's dimension. This is a particularly remarkable feature of the approach.

- The approach involves a dynamic extension of the initial system, its dimension being equal to the sum of the dimensions of variables $x$ (or $z$) and $\theta$. This extension is used to formulate a problem of stabilization (of the vector $z$) easily solved by applying classical techniques (state feedback linearization, followed by closed-loop pole placement, for example).

- State feedbacks obtained via this approach are everywhere differentiable (and even analytic) so that numerical evaluation of the control variables should not present difficulties – by contrast with discontinuous or continuous but non-Lipschitz controls. More precisely, numerical problems will not appear unless the control expression tends to become singular, when the matrix $B(f_\epsilon(\theta), \theta))$ itself tends to become singular by letting the upperbound of the norm of $f_\epsilon(\theta)$ tend to zero.

- The approach provides a "uniform" solution to the problem of tracking a reference vehicle by a nonholonomic wheeled vehicle. By "uniform" we mean that the state feedback expression, and the performance granted by analysis, do not involve any condition upon the motion of the reference vehicle. It is not even required that the reference vehicle be subjected to nonholonomic constraints similar to those upon the controlled vehicle.

- The control characteristics, beyond the traditional criteria of rates of convergence and damping – which, in the present case and contary to other control methods, are easily tunable – much depend upon the choice of the transverse function $f_\epsilon$. Study of this choice will logically be an important issue for future developments of the approach. For instance, the determination of parameters $\epsilon_1$ et $\epsilon_2$ involved in the expression of the function defined by (22) could be explicited in relation to the satisfaction of objective criteria (which remain to be defined).

## 9. Bibliographical landmarks

Many research texts are devoted to mobile robot control. Concerning references in the form of surveys, let us mention [5], [21], and [52], which include chapters about modeling and control of mobile robots. Let us also mention the forthcoming publication of [22], a text written in French. Next references will be cited by following the organization of the article.

A detailed study of kinematic and dynamic models of various types of mobile robots is in [4].

Concerning controllability aspects, the reader will find basic definitions and general properties of controllability in classical books about nonlinear control systems like [17, 40].

The use of homogeneity in Control Theory has a long history. Within the framework of interest to us here, the reader may consult the surveys [16, 18]. The use of the chained form to represent mobile robot equations was proposed in [37], and generalized in [47].

The trajectory stabilization problem – with the point of view of tracking a reference vehicle – is treated in surveys like [5] and [21], and also in many conference and journal papers. Several authors have also addressed this problem via dynamic feedback linearization techniques. On this subject, one may consult [8], [14], as well as [5, Chap. 8]. The path following problem has perhaps been addressed before by researchers in mobile robotics. Among pioneering works in this domain, one may cite [12] and [39]. Ideas presented in the present article are mostly based on [45] and [46]. In particular, the latter reference describes solutions to the path following for a $n$-dimensional chained system in details.

Numerous articles about stabilization of fixed configurations have been published recently. The reference [19] provides a survey of state feedback control techniques elaborated to this purpose, as well as an extensive list of references. The conference article [29] is also a survey. It is entirely devoted to the class of time-varying feedbacks, within the general framework of nonlinear control systems. The first result concerning the application of time-varying feedback to the stabilization of nonholonomic systems, and more generally to systems which cannot be stabilized via continuous pure-state feedback, has been given in [44]. Concerning the design of differentiable time-varying stabilizers, the reader may also consult [41, 46, 50]. As for the case of stabilizers which are only continuous but ensure exponential convergence, one may refer to [27, 30], where general methods for driftless control systems are presented. The control law (19) is taken from [33] where the focus is on the specific class of chained systems. Hybrid continuous/discrete feedbacks may be found in [2] and [48] for example. The control law (21) is taken from [31]. Discontinuous feedbacks are not addressed here, but the interested reader will find examples of such control laws in [6].

Nonexistence of universal trajectory asymptotic stabilizers for chained systems is proved in [24, Ch. 5].

Concerning the control approach presented in Section 8, the reader may consult [32] as for general theoretical results, and [34] for its application to mobile robot control – based on the chained system representation. The article [13], about the control of a unicycle-type mobile robot, may also be seen as an application of this approach.

Clearly, many important problems have not been addressed in the present article. We give below a – very incomplete – list of bibliographical pointers to some of these problems.

An important practical problem concerns path planning and the associated steering (open-loop) control problem. Among already old but still instructive studies on the subject, let us cite [20, 23], with the second reference being a collective work. For a more recent exposition, the reader may consult [21]. Concerning steering control more specifically, one may consult some articles in [23], as well as [38], and the literature devoted to differential *flatness* [15, 42],...

Sensor-based control of mobile robots has been an active domain of research for years. Vision-based road following is an example of a subject which has already motivated numerous studies and experimentations, and is still likely to feed research for years to come due to the multiple, complex, and multidisciplinary aspects involved in it. Most of published vision-based control schemes for mobile robots involve classical control techniques (linear approximation, pole placement, Linear Quadratic control,...) – in accordance with, and as an illustration of, explanations given in the present article about partial state stabilization and trajectory stabilization. However, this indicates also that the problem of stabilizing a nonholonomic mobile robot fixed configuration via a sensor-based control approach has little been addressed so far. Nevertheless, an attempt on this subject, mixing time-varying feedback and vision-based control techniques, is described in [51].

Finally, it is worth mentioning that numerous results about mobile robots pulling trailors apply only to the case when the hitch-point of each trailor is located on the rear axle of the preceding vehicle – in connection with the possibility of representing these systems by chained systems. For the other cases, many problems are still open. A few feedback control studies in this domain are nonetheless available. Let us cite, for instance, [1] and the thesis work [24].

## Appendix. Homogeneous systems: definitions and basic properties

We give in this appendix a few definitions and properties about homogeneity, either explicitly or implicitly used in Sections 4 and 6. For more details, we refer the reader to [16].

Given a *weight vector* $r = (r_1, \dots, r_n)$ with $0 < r_i \in \mathbb{R}$, one defines, for any $\lambda > 0$, a *dilation* associated with this weight vector as the mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ given by

$$\forall x \in \mathbb{R}^n\,, \qquad \Delta_\lambda^r x = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n)$$

A function $f \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R})$ is said to be *homogeneous* of degree $d$ with respect to (w.r.t.) the dilation $\Delta_\lambda^r$ if

$$\forall \lambda \geqslant 0,\ \forall x \in \mathbb{R}^n\,, \qquad f(\Delta_\lambda^r x) = \lambda^d f(x)$$

A vector field $b$ on $\mathbb{R}^n$ – $b \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n)$ – is said to be *homogeneous* of degree $\tau$ w.r.t. the dilation $\Delta_\lambda^r$, if for every integer $k \in \{1, \dots, n\}$, the $k$-th component $b_k$ of $b$ is a homogeneous function of degree $\tau + r_k$. By extension, a control system (1) is homogeneous of degree $\tau$ w.r.t. the dilation $\Delta_\lambda^r$ if every vector field $b_i$ is homogeneous of degree $\tau$ w.r.t. this dilation.

The following properties generalize some of the properties of linear systems and illustrate the usefulness of homogeneous systems

1. Every locally controllable system (1) can be locally approximated by a homogeneous system itself locally controllable. The latter system is then called a *homogeneous approximation* of system (1) .

2. Consider the differential equation $\dot{x} = f(x,t)$, with $f(.,t)$ a vector field homogeneous of degree zero (w.r.t. some arbitrary dilation) for every $t$, and assume that $f$ is periodic w.r.t. $t$. If $x = 0$ is a locally stable equilibrium point of this equation, then it is globally exponentially stable – in the sense that there exist two positive numbers $k$ and $\alpha$ such that along any solution to this system

$$\forall t \geqslant t_0 \ : \quad \rho(x(t)) \ \leqslant k\rho(x(t_0))exp(-\alpha t) \tag{30}$$

   with $\rho$ denoting a *homogeneous norm* – a proper positive function homogeneous of degree one.

3. Let

$$\dot{x} = \sum_{i=1}^{m} \widetilde{b}(x)u_i \tag{31}$$

   denote a homogeneous approximation of the system (1) – w.r.t. a dilatation $\Delta_\lambda^r$. Let $\tau_1, \ldots, \tau_m$ denote the respective degrees of homogeneity of the vector fields $\widetilde{b}_1, \ldots \widetilde{b}_m$. Let $u = (u_1, \ldots, u_m)$ be a time-varying state feedback – i.e. $u(x,t)$ –, periodic in $t$, and such that, for any $t$, the function $u_i(.,t)$ is homogeneous of degree $-\tau_i$ w.r.t. $\Delta_\lambda^r$. *If* the origin of the closed-loop system obtained by applying this feedback to (31) is locally asymptotically stable, *then* all conditions of Property 2 above are satisfied, thus implying that the origin is in fact a globally exponentially stable equilibrium point. Moreover, the origin of the closed-loop system obtained by applying the same feedback to the original system (1) is locally exponentially stable.

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